NUMERICAL PROPERTIES OF AMPLENESS

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Definition 0.1. Two Cartier divisors D_1, D_2 are **numerically equivalent** ($D_1 \equiv_{\text{num}} D_2$) if $D_1 \cdot C = D_2 \cdot C$ for all irreducible curves in X.

A divisor or line bundle is numerically trivial if it is numerically equivalent to 0.

Let $\text{Num}(X) \subset \text{Div}(X)$ be the subgroup of numerically trivial divisors.

Definition 0.2. The Neron-Severi group of X is

$$N^1(X) = \text{Div}(X)/\text{Num}(X).$$

Fact: $N^1(X)$ is a free abelian group of finite rank.

Definition 0.3. The rank of $N^1(X)$ is called the **Picard number** of X ($\rho(X)$).

Theorem 0.4. *(Nakai-Moishezon-Kleiman criterion for ampleness) Let L be a line bundle on a projective scheme X. L is ample $\Leftrightarrow \int_V c_1(L)^{\dim V} > 0$ for any positive dimensional irreducible subvariety V of X.*

The following is a history of Nakai-Moishezon criterion.
Nakai: for smooth surfaces (1960)
Moishezon: for nonsingular varieties of higher dimension (1961), singular varieties (1962)
Nakai: for projective schemes (1963)
Fedorchuk, Smyth: for algebraic space (2011) (with an additional condition)

Corollary 0.5. *(Numerical nature of ampleness) If $D_1, D_2 \in \text{Div}(X)$ such that $D_1 \equiv_{num} D_2$ on a projective scheme, then D_1 ample $\Leftrightarrow$ D_2 ample.*

Remark 0.6. $\delta \in N^1(X)$ is ample if it represented by an ample divisor.

Corollary 0.7. *(Finite pullblacks) Let $f : Y \to X$ finite and surjective. Let L be a line bundle on X. If f^*L is ample on Y then L is ample on X.*

Proof. Let $V \subset X$ be an irreducible subvariety. Since f is surjective, there is $W \subset Y$ which maps finitely onto V. By projection formula, we have

$$0 < \int_W c_1(f^*L)^{\dim W} = \deg(W \to V) \int_V c_1(L)^{\dim V}.$$

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Proof. (of Theorem 0.4) $\Leftarrow$ Relies heavily on the following result:

Proposition 0.8. *(Corollary 1.2.15) Suppose L globally generated and $\phi : X \to \mathbb{P}H^0(X, L)$ the corresponding morphism. Then L is ample $\Leftrightarrow \int_C c_1(L) > 0$ for every irreducible curve C.*

By Proposition 1.2.16, we may suppose that X is reduced and irreducible. The result for $\dim X = 1$ is clear.

Proceed by induction on $\dim X$. Let $n = \dim X$. Assume that theorem holds for all schemes of dimension $\leq n-1$.

Write $L = \mathcal{O}_X(D)$. Want to show: $H^0(X, \mathcal{O}_X(mD)) \neq 0$, for $m \gg 0$. By asymptotic Riemann-Roch,

$$\chi(X, \mathcal{O}_X(mD)) = m^n \frac{(D^n)}{n!} + O(m^{n-1}).$$

Note that $D^n = \int_X c_1(L)^n > 0$ by assumption.

Write $D \equiv_{lin} A - B$, A, B very ample divisors.

$$0 \to \mathcal{O}_X(mD - B) \to \mathcal{O}_X((m+1)D) \to cO_A((m+1)D) \to 0$$
$$0 \to \mathcal{O}_X(mD - B) \to \mathcal{O}_X(mD) \to \mathcal{O}_B(mD) \to 0$$

By induction, $\mathcal{O}_A(D)$ and $\mathcal{O}_B(D)$ are ample. Serre vanishing $\Rightarrow$ higher cohomology of $\mathcal{O}_A((m+1)D)$, $\mathcal{O}_B(mD)$ will vanish for $m \gg 0$. Thus

$$h^i(\mathcal{O}_X(mD)) = h^i(\mathcal{O}_X(mD - B)) = h^i(\mathcal{O}_X((m+1)D)), \quad i \geq 2.$$

Therefore,

$$\chi(\mathcal{O}_X(mD)) = h^0(\mathcal{O}_X(mD)) - h^1(\mathcal{O}_X(mD)) + C,$$

and

$$h^0(\mathcal{O}_X(mD)) = h^1(\mathcal{O}_X(mD)) + m^n \frac{D^n}{n!} + O(m^{n-1})$$

so $h^0(\mathcal{O}_X(mD)) \neq 0$ for $m \gg 0$. By replacing D by mD, we may assume that D is effective.

Using effectivity of D, D will be globally generated away $\mathrm{supp}(D)$. Indeed, mD is globally generated. It is sufficient to show the global generation on $\mathrm{supp}(D)$. Note that $\mathcal{O}_D(mD)$ is globally generated by ampleness of D on D (from induction hypothesis). From

$$0 \to \mathcal{O}_X((m-1)D) \overset{\cdot D}{\to} \mathcal{O}_X(mD) \to \mathcal{O}_D(mD) \to 0,$$
$$H^1(\mathcal{O}_X((m-1)D) \to H^1(\mathcal{O}_X(mD))$$

is surjective for $m \gg 0$ ($H^1(\mathcal{O}_D(mD)) = 0$ from Serre's vanishing). Thus $H^1(\mathcal{O}_X((m-1)D) \to H^1(\mathcal{O}_X(mD))$ is isomorphism for $m \gg 0$. So $H^0(\mathcal{O}_X(mD)) \to H^0(\mathcal{O}_D(mD))$ is surjective for that m. ☐