AMPLENESS

JOE TENINI, HAN-BOM MOON

Definition 0.1. Let X be a complete scheme and L be a line bundle. L is **very ample** if there is an embedding $X \hookrightarrow \mathbb{P}^N$ such that $L = \mathcal{O}_{\mathbb{P}^N}(1)|_X$. L is **ample** if there is $m \in \mathbb{N}$ such that L^m is very ample.

Example 0.2. (1) For a curve X, L is ample $\Leftrightarrow \deg L > 0$.
(2) For a smooth curve X, D is very ample $\Leftrightarrow \forall P, Q \in X, \dim |D-P-Q| = \dim |D| - 2$.
(3) If X is projective and $\mathrm{Pic} X = \mathbb{Z} \Rightarrow$ effectiveness implies ampleness.

Remark 0.3. For n-dimensional variety X and n ample divisors $D_1, \cdots, D_n$, then $D_1 \cdot \cdots \cdot D_n > 0$.

Cohomological characterization of ampleness.

Theorem 0.4. *(Cartan-Serre-Grothendieck) TFAE:*

(1) L is ample;
(2) For a coherent sheaf F, $\exists m_1(F) \in \mathbb{N}$ such that $H^i(X, F \otimes L^m) = 0$ for all $i > 0, m \geq m_1(F)$;
(3) For a coherent sheaf F, $\exists m_2(F) \in \mathbb{N}$ such that $F \otimes L^m$ is globally generated for $m \geq m_2(F)$.
(4) $\exists m_3 \in \mathbb{N}$ such that L^m is very ample for $m \geq m_3$.

Proof. ($1 \Rightarrow 2$)

Assume that L is very ample and $X \hookrightarrow \mathbb{P}^r$ is the embedding given by L. One can regard F as a sheaf on $\mathbb{P}^r$ (extending by 0). It suffices to show that $H^i(F(m)) = 0$ for $i > 0, m >> 0$. See [Har77, Theorem III.5.2].

($2 \Rightarrow 3$)

Fix $x \in X$, let $m_x \subset \mathcal{O}_X$ be the ideal sheaf of x. By (2), $H^1(X, m_x F \otimes L^m) = 0$ for $m \geq m_2$.

$$0 \to m_x F \to F \to F/m_x F \to 0$$

$$0 \to m_x F \otimes L^m \to F \otimes L^m \to F/m_x F \otimes L^m \to 0$$

$$0 \to H^0(m_x F \otimes L^m) \to H^0(F \otimes L^m) \to H^0(F/m_x F) \to 0$$

We can take common m by quasi-compactness of X. Thus it is globally generated.

($3 \Rightarrow 4$)

Date: November 9, 2012.

$\exists$ positive integer m_2 such that L^m is globally generated for all $m \geq m_2$. Let $\varphi_m : X \to \mathbb{P}H^0(X, L^m)$. Define $U_m = \{y \in X | L^m \otimes m_y$ is globally generated$\}$. This is an open subset and $U_m \subset U_{m+1}$. Since X is noetherian, $X = U_m$ for some m.

The claim is it gives the proof of separation of points and separation of tangent vectors. Pick $x, y \in X$. Then since $L^m \otimes m_y$ is globally generated, there is a section $s \in H^0(L^m \otimes m_y)$ such that $s(x) \neq 0$. By the definition of $L^m \otimes m_y$, it gives a section s such that $s(x) \neq 0$ and $s(y) = 0$. Therefore we obtain the separation of points. Moreover, from the global generation, the restriction morphism $H^0(X, L^m \otimes m_y) \to H^0(X, L^m \otimes m_y|_y) \cong H^0(L^m \otimes m_y/m_y^2)$ is surjective. Therefore for any tangent vector $v \in \mathrm{Hom}(m_y/m_y^2, \mathbb{C})$, there is a section $s \in H^0(X, L^m \otimes m_y)$ such that $s|_y(v) \neq 0$. It gives the separation of tangent vectors.

($4 \Rightarrow 1$)

This is just a definition. $\square$

References

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52. 1