Homework 2 Model Solution

Section 12.5.

12.5.4 Find a vector equation and parametric equations for the line through the point $(0, 14, -10)$ and parallel to the line $x = -1 + 2t$, $y = 6 - 3t$, $z = 3 + 9t$.

A point on the line: $(0, 14, -10)$ or $14\mathbf{j} - 10\mathbf{k}$

A direction vector: $\langle 2, -3, 9\rangle$ or $2\mathbf{i} - 3\mathbf{j} + 9\mathbf{k}$

Vector equation: $\mathbf{r}(t) = (14\mathbf{j} - 10\mathbf{k}) + t(2\mathbf{i} - 3\mathbf{j} + 9\mathbf{k}) = 2t\mathbf{i} + (14 - 3t)\mathbf{j} + (-10 + 9t)\mathbf{k}$

Parametric equations: $x = 2t,\ y = 14 - 3t,\ z = -10 + 9t$

12.5.10 Find parametric equations and symmetric equations for the line through $(2, 1, 0)$ and perpendicular to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$.

A point on the line: $(2, 1, 0)$

A direction vector:

$$(\mathbf{i} + \mathbf{j}) \times (\mathbf{j} + \mathbf{k}) = \langle 1, 1, 0\rangle \times \langle 0, 1, 1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

Vector equation: $\mathbf{r}(t) = \langle 2, 1, 0\rangle + t\langle 1, -1, 1\rangle = \langle 2 + t, 1 - t, t\rangle$

Parametric equations: $x = 2 + t,\ y = 1 - t,\ z = t$

Symmetric equations: $\dfrac{x-2}{1} = \dfrac{y-1}{-1} = \dfrac{z}{1}$ or $x - 2 = 1 - y = z$

12.5.15 (a) Find symmetric equations for the line that passes through the point $(1, -5, 6)$ and is parallel to the vector $\langle -1, 2, -3\rangle$.

Symmetric equations: $\dfrac{x-1}{-1} = \dfrac{y+5}{2} = \dfrac{z-6}{-3}$

(b) Find the points in which the required line in part (a) intersects the coordinate planes.

$$z = 0 \Rightarrow \frac{x-1}{-1} = \frac{y+5}{2} = \frac{-6}{-3} = 2 \Rightarrow x = -1, y = -1$$

Intersection with xy-plane: $(-1, -1, 0)$

$$y = 0 \Rightarrow \frac{x-1}{-1} = \frac{z-6}{-3} = \frac{5}{2} \Rightarrow x = -\frac{3}{2}, z = -\frac{3}{2}$$

Intersection with xz-plane: $(-\frac{3}{2}, 0, -\frac{3}{2})$

$$x = 0 \Rightarrow \frac{y+5}{2} = \frac{z-6}{-3} = \frac{-1}{-1} = 1 \Rightarrow y = -3, z = 3$$

Intersection with yz-plane: $(0, -3, 3)$

12.5.20 Determine whether the lines

$$L_1 : x = 5 - 12t, \quad y = 3 + 9t, \quad 1 - 3t$$

and

$$L_2 : x = 3 + 8s, \quad y = -6s, \quad z = 7 + 2s$$

are parallel, skew, or intersecting. If they intersect, find the point of intersection.

A direction vector of L_1: $\mathbf{v}_1 = \langle -12, 9, -3 \rangle$

A direction vector of L_2: $\mathbf{v}_2 = \langle 8, -6, 2 \rangle$

$\mathbf{v}_1 = -\frac{3}{2}\mathbf{v}_2 \Rightarrow$ Two lines are parallel.

Moreover, $(3, 0, 7)$ is on L_2 ($s = 0$). But $5 - 12t = 3, 3 + 9t = 0, 1 - 3t = 7$ does not have a common solution, so $(3, 0, 7)$ is not on L_1. Therefore L_1 and L_2 are parallel but distinct lines.

12.5.24 Find an equation of the plane through the point $(5, 3, 5)$ and with normal vector $2\mathbf{i} + \mathbf{j} - \mathbf{k}$.

An equation of the plane:

$$2(x - 5) + (y - 3) + (-1)(z - 5) = 0 \Rightarrow 2x - 10 + y - 3 - z + 5 = 0$$

$$\Rightarrow 2x + y - z - 8 = 0$$

12.5.26 Find an equation of the plane through the point $(2, 0, 1)$ and perpendicular to the line $x = 3t$, $y = 2 - t$, $z = 3 + 4t$.

A point on the plane: $(2, 0, 1)$

A normal vector to the plane: $3\mathbf{i} - \mathbf{j} + 4\mathbf{k}$

An equation of the plane:

$$3(x - 2) - (y - 0) + 4(z - 1) = 0 \Rightarrow 3x - 6 - y + 4z - 4 = 0$$

$$\Rightarrow 3z - y + 4z - 10 = 0$$

12.5.31 Find an equation of the plane through the points $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$.

A point on the plane: $(0, 1, 1)$

$$\mathbf{v}_1 = \langle 1, 0, 1 \rangle - \langle 0, 1, 1 \rangle = \langle 1, -1, 0 \rangle, \mathbf{v}_2 = \langle 1, 1, 0 \rangle - \langle 0, 1, 1 \rangle = \langle 1, 0, -1 \rangle$$

A normal vector to the plane:

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = \langle 1, 1, 1 \rangle$$

An equation of the plane:

$$1(x - 0) + 1(y - 1) + 1(z - 1) = 0 \Rightarrow x + y + z - 2 = 0$$

12.5.39 Find an equation of the plane that passes through the point $(1, 5, 1)$ and is perpendicular to the planes $2x + y - 2z = 2$ and $x + 3z = 4$.

A point on the plane: $(1, 5, 1)$

A normal vector to the plane:

$$\langle 2, 1, -2\rangle \times \langle 1, 0, 3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -2 \\ 1 & 0 & 3 \end{vmatrix} = 3\mathbf{i} - 8\mathbf{j} - \mathbf{k}$$

An equation for the plane:

$$3(x-1) - 8(y-5) - (z-1) = 0 \Rightarrow 3x - 3 - 8y + 40 - z + 1 = 0$$

$$\Rightarrow 3x - 8y - z + 38 = 0$$

12.5.46 Find the point at which the line

$$x = 1 + 2t, \quad y = 4t, \quad z = 2 - 3t$$

intersects the plane $x + 2y - z + 1 = 0$.

$$(1 + 2t) + 2(4t) - (2 - 3t) + 1 = 0 \Rightarrow 1 + 2t + 8t - 2 + 3t + 1 = 0$$

$$\Rightarrow 13t = 0 \Rightarrow t = 0$$

$$\Rightarrow x = 1, y = 0, z = 2$$

Intersection point: $(1, 0, 2)$

12.5.55 Determine whether the planes

$$x = 4y - 2z, \quad 8y = 1 + 2x + 4z$$

are parallel, perpendicular, or neither. If neither, find the angle between them.

$$x = 4y - 2z \Rightarrow x - 4y + 2z = 0$$

A normal vector of the first plane: $\mathbf{n}_1 = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$

$$8y = 1 + 2x + 4z \Rightarrow 2x - 8y + 4z + 1 = 0$$

A normal vector of the second plane: $\mathbf{n}_2 = 2\mathbf{i} - 8\mathbf{j} + 4\mathbf{k}$

$\mathbf{n}_2 = 2\mathbf{n}_1 \Rightarrow$ Two planes are parallel.

12.5.59 Find symmetric equations for the line of intersection of the planes

$$5x - 2y - 2z = 1, 4x + y + z = 6.$$

A point on the line:

$$z = 0 \Rightarrow 5x - 2y = 1, 4x + y = 6 \Rightarrow 5x - 2y = 1, y = 6 - 4x \Rightarrow 5x - 2(6 - 4x) = 1$$

$$\Rightarrow 13x = 13 \Rightarrow x = 1 \Rightarrow y = 2$$

$(1, 2, 0)$ is on the line.

A direction vector:

$$(5\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \times (4\mathbf{i} + \mathbf{j} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -2 & -2 \\ 4 & 1 & 1 \end{vmatrix} = -13\mathbf{j} + 13\mathbf{k}$$

Symmetric equations for the ilne:

$$x = 1, \frac{y-2}{-13} = \frac{z-0}{13} \text{ or } x = 1, 2 - y = z$$

12.5.72 Find the distance from the point $(-6, 3, 5)$ to the plane $x - 2y - 4z = 8$.

$$x - 2y - 4z = 8 \Leftrightarrow x - 2y - 4z - 8 = 0$$
$$\text{distance } = \frac{|-6 - 2 \cdot 3 - 4 \cdot 5 - 8|}{\sqrt{1^2 + (-2)^2 + (-4)^2}} = \frac{40}{\sqrt{21}}$$

12.5.80 Let L_1 be the line through the points $(1, 2, 6)$ and $(2, 4, 8)$. Let L_2 be the line of intersection of the planes π_1 and π_2 where π_1 is the plane $x - y + 2z + 1 = 0$ and π_2 is the plane through the points $(3, 2, -1)$, $(0, 0, 1)$, and $(1, 2, 1)$. Calculate the distance between L_1 and L_2.

A point on L_1: $(1, 2, 6)$

Direction vector of L_1: $\mathbf{v}_1 = \langle 2, 4, 8\rangle - \langle 1, 2, 6\rangle = \langle 1, 2, 2\rangle$

A point on π_2: $(0, 0, 1)$

Normal vector of π_2:

$$(\langle 3, 2, -1\rangle - \langle 0, 0, 1\rangle) \times (\langle 1, 2, 1\rangle - \langle 0, 0, 1\rangle) = \langle 3, 2, -2\rangle \times \langle 1, 2, 0\rangle$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -2 \\ 1 & 2 & 0 \end{vmatrix} = \langle 4, -2, 4\rangle$$

An equation for π_2:

$$4(x - 0) - 2(y - 0) + 4(z - 1) = 0 \Rightarrow 4x - 2y + 4z - 4 = 0$$

A point on L_2, which is the intersection of π_1 and π_2:

$$z = 0 \Rightarrow x - y + 1 = 0, 4x - 2y - 4 = 0 \Rightarrow x = 3, y = 4$$

$$(x, y, z) = (3, 4, 0)$$

Direction vector of L_2:

$$\mathbf{v}_2 = \langle 1, -1, 2\rangle \times \langle 4, -2, 4\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ 4 & -2 & 4 \end{vmatrix} = \langle 0, 4, 2\rangle$$

$$\mathbf{v}_1 \times \mathbf{v}_2 = \langle 1, 2, 2\rangle \times \langle 0, 4, 2\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ 0 & 4 & 2 \end{vmatrix} = \langle -4, -2, 4\rangle$$

P_1: a plane containing L_1 and having $\mathbf{v}_1 \times \mathbf{v}_2$ as a normal vector

$$-4(x-1) - 2(y-2) + 4(z-6) = 0 \Rightarrow -4x + 4 - 2y + 4 + 4z - 24 = 0$$

$$\Rightarrow -4x - 2y + 4z - 16 = 0$$

P_2: a plane containing L_2 and having $\mathbf{v}_1 \times \mathbf{v}_2$ as a normal vector

$$-4(x-3) - 2(y-4) + 4(z-0) = 0 \Rightarrow -4x + 12 - 2y + 8 + 4z = 0$$

$$\Rightarrow -4x - 2y + 4z + 20 = 0$$

distance between L_1 and L_2 = distance between P_1 and P_2 = distance from a point $(3, 4, 0)$ on P_2 to P_1

$$\text{distance } = \frac{|-4\cdot 3 - 2\cdot 4 + 4\cdot 0 - 16|}{\sqrt{(-4)^2 + (-2)^2 + 4^2}} = \frac{36}{\sqrt{36}} = 6$$