Homework 10 Solution
Chapter 9.

1. Let $H = \{(1), (12)\}$. Is $H$ normal in $S_3$?

$$(13)H = \{(13)(1), (13)(12)\} = \{(13), (123)\}$$

$$H(13) = \{(1)(13), (12)(13)\} = \{(13), (132)\}$$

Because $(13)H \neq H(13)$, $H$ is not a normal subgroup of $S_3$.

8. Viewing $\langle 3 \rangle$ and $\langle 12 \rangle$ as subgroups of $\mathbb{Z}$, prove that $\langle 3 \rangle/\langle 12 \rangle$ is isomorphic to $\mathbb{Z}_4$. Similarly, prove that $\langle 8 \rangle/\langle 48 \rangle$ is isomorphic to $\mathbb{Z}_6$. Generalize to arbitrary integers $k$ and $n$.

I will prove the general formula: For any positive integers $k$ and $n$, two groups $\langle k \rangle/\langle kn \rangle$ and $\mathbb{Z}_n$ are isomorphic.

Sol 1. Because $\langle k \rangle$ is cyclic, all elements in $\langle k \rangle$ is of the form $mk$ for $m \in \mathbb{Z}$. So all elements in $\langle k \rangle/\langle kn \rangle$ is of the form $mk + \langle kn \rangle = m(k + \langle kn \rangle)$. Therefore $\langle k \rangle/\langle kn \rangle$ is cyclic and it is generated by $k + \langle kn \rangle$. So it suffices to check the order of $k + \langle kn \rangle$. Note that $n(k + \langle kn \rangle) = nk + \langle kn \rangle = \langle kn \rangle$ and for $0 < m < n$, $m(k + \langle kn \rangle) \neq \langle kn \rangle$. Therefore $|k + \langle kn \rangle| = n$ and $\langle k \rangle/\langle kn \rangle \cong \mathbb{Z}_n$.

Sol 2. Note that an element in $\langle k \rangle/\langle kn \rangle$ is of the form $mk + \langle kn \rangle$ for some $m \in \mathbb{Z}$. Define a map $\phi : \langle k \rangle/\langle kn \rangle \to \mathbb{Z}_n$ as $\phi(mk + \langle kn \rangle) = m \mod n$.

Step 0. $\phi$ is well-defined.

If $m_1k + \langle kn \rangle = m_2k + \langle kn \rangle$, then $-m_2k + m_1k + \langle kn \rangle = \langle kn \rangle$. So $(m_1 - m_2)k = -m_2k + m_1k \in \langle kn \rangle$. Therefore $m_1 - m_2$ is a multiple of $n$ so $m_1 \mod n = m_2 \mod n$. This implies that $\phi(m_1k + \langle kn \rangle) = \phi(m_2k + \langle kn \rangle)$.

Step 1. $\phi$ is one-to-one.

If $\phi(m_1k + \langle kn \rangle) = \phi(m_2k + \langle kn \rangle)$, then $m_1 \mod n = m_2 \mod n$. Therefore $n|m_1 - m_2$ and $nk|m_1k - m_2k$. So $m_1k - m_2k \in \langle kn \rangle$, and $m_1k - m_2k + \langle nk \rangle = \langle nk \rangle$. So $m_1k + \langle nk \rangle = m_2k + \langle nk \rangle$.

Step 2. $\phi$ is onto.

Obviously, for $m \in \mathbb{Z}_n$, $\phi(mk + \langle kn \rangle) = m \mod n = m$.

Step 3. $\phi$ has the operation preserving property.

$$\phi(m_1k + \langle kn \rangle)\phi(m_2k + \langle kn \rangle) = m_1 \mod n + m_2 \mod n = m_1 + m_2 \mod n$$

$$= \phi((m_1 + m_2)k + \langle kn \rangle)$$

$$= \phi((m_1k + \langle kn \rangle) + (m_2k + \langle kn \rangle))$$.

Therefore $\phi$ is an isomorphism.
9. Prove that if \( H \) has index 2 in \( G \), then \( H \) is normal in \( G \).

Because \( H \) has index 2, there are exactly two left cosets (say \( \{ H, aH \} \)) and two right cosets (\( \{ H, Hb \} \)). Note that the disjoint union of \( H \) and \( aH \) is \( G \). Also the disjoint union of \( H \) and \( Hb \) is \( G \). So \( aH = G - H = Hb \).

If \( x \in H \), then \( xH = H = Hx \). If \( x \notin H \), then \( xH = G - H = Hx \). So in any cases, the left coset is equal to the right coset. Therefore \( H \) is normal.

11. Let \( G = \mathbb{Z}_4 \oplus U(4), \) \( H = \langle (2, 3) \rangle \), and \( K = \langle (2, 1) \rangle \). Show that \( G/H \) is not isomorphic to \( G/K \). (This shows that \( H \approx K \) does not imply that \( G/H \approx G/K \).)

In \( G/H \), \( ((1, 3)H)^2 = (1 + 1, 3^2)H = (2, 1)H, \) \( ((1, 3)H)^3 = (1 + 1 + 1, 3^3)H = (3, 3)H, \) \( ((1, 3)H)^4 = (1 + 1 + 1 + 1, 3^4)H = (0, 1)H = H \) (Note that \( (0, 1) \) is the identity of \( G \)). So \( |(1, 3)H| = 4 \) in \( G/H \).

On the other hand, in \( G/K \), for any element \( (a, b)H \), \( ((a, b)H)^2 = (a + a, b^2)H = (2a, b^2)H \). But in \( U(4) = \{1, 3\} \), all squares are 1, so \( (2a, b^2)H = (2a, 1)H = \mathbb{Z}_2 \). Therefore \( |(a, b)H| \leq 2 \).

So \( G/H \not\approx G/K \).

12. Prove that a factor group of a cyclic group is cyclic.

Let \( G = \langle a \rangle \) and \( H \trianglelefteq G \). Then any element in \( G/H \) is of the form \( a^kH \), which is \( (aH)^k \) for some \( k \in \mathbb{Z} \). Therefore \( G/H = \langle aH \rangle \) and it is cyclic.

14. What is the order of the element \( 14 + \langle 8 \rangle \) in the factor group \( \mathbb{Z}_{24}/\langle 8 \rangle \)?

\[
14 + \langle 8 \rangle = 6 + \langle 8 \rangle \\
2 \cdot (6 + \langle 8 \rangle) = 12 + \langle 8 \rangle = 4 + \langle 8 \rangle \\
3 \cdot (6 + \langle 8 \rangle) = 18 + \langle 8 \rangle = 2 + \langle 8 \rangle \\
4 \cdot (6 + \langle 8 \rangle) = 24 + \langle 8 \rangle = \langle 8 \rangle
\]

So \( |14 + \langle 8 \rangle| = 4 \).

Can you generalize it? Answer: In \( \mathbb{Z}_n, |a + \langle b \rangle| = \text{lcm}(a, \text{gcd}(n, b))/a \).

21. Prove that an Abelian group of order 33 is cyclic.

Let \( G \) be an Abelian group of order 33. There is \( a \in G \) with \( |a| = 11 \) and \( b \in G \) with \( |b| = 3 \).

Sol 1. Because \( G \) is Abelian, \( (ab)^i = a^i b^i \). The order \( |ab| \) is one of 1, 3, 11, 33. If \( |ab| = 1 \), then \( ab = e \) and \( b = a^{-1} \). So we obtain a contradiction \( |a| = |b| \). If \( |ab| = 3 \), then \( e = (ab)^3 = a^3 b^3 = a^3 \). This is impossible because \( |a| = 11 \). If \( |ab| = 11 \), then \( e = (ab)^{11} = a^{11} b^{11} = b^{11} \). From \( b^{11} = e \), we have \( b^2 = e \), which is impossible too. Therefore \( |ab| = 33 \) and \( G = \langle ab \rangle \).

Sol 2. Let \( H = \langle a \rangle \) and \( K = \langle b \rangle \). \( H \cap K = \{ e \} \) because \( |H \cap K| \) is a common divisor of \( |H| = 11 \) and \( |K| = 3 \). Because \( |H \cap K| = [H||K]/|H \cap K| = 11 \cdot 3/1 = 33 = |G| \), \( G = H \times K \approx H \oplus K \approx \mathbb{Z}_{11} \oplus \mathbb{Z}_3 \approx \mathbb{Z}_{33} \).
22. Determine the order of \((\mathbb{Z} \oplus \mathbb{Z})/\langle(2, 2)\rangle\). Is the group cyclic?

For \((1, 0) \in \mathbb{Z} \oplus \mathbb{Z}\), \(m(1, 0) = (m, 0) \notin \langle(2, 2)\rangle\) for all \(m > 0\). This implies that \(m((1, 0) + \langle(2, 2)\rangle) \neq \langle(2, 2)\rangle\) for any \(m > 0\). So \(|(1, 0) + \langle(2, 2)\rangle| = \infty\), and 
\\
\[2((1, 1) + \langle(2, 2)\rangle) = (2, 2) + \langle(2, 2)\rangle = \langle(2, 2)\rangle\]
\\
So \(|(1, 1) + \langle(2, 2)\rangle| = 2\). On the infinite cyclic group \(\mathbb{Z}\), except the identity, there is no element with finite order. Therefore \((\mathbb{Z} \oplus \mathbb{Z})/\langle(2, 2)\rangle\) is not cyclic.

24. The group \((\mathbb{Z}_4 \oplus \mathbb{Z}_{12})/\langle(2, 2)\rangle\) is isomorphic to one of \(\mathbb{Z}_8\), \(\mathbb{Z}_4 \oplus \mathbb{Z}_2\), or \(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2\). Determine which one by elimination.

Note that \((\langle 2,2 \rangle) = \{(0,0),(2,2),(0,4),(2,6),(0,8),(2,10)\}\). For \((0,1) \in \mathbb{Z}_4 \oplus \mathbb{Z}_{12}\), \(k(0,1) = (0,k) \in \langle(2,2)\rangle\) only if \(4|k\). So \(|(0,1)+\langle(2,2)\rangle| = 4\). So it is not isomorphic to \(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2\) where all nonidentity elements have order 2.

Furthermore, an element \((a,b) \in \mathbb{Z}_4 \oplus \mathbb{Z}_{12}\) has order \(lcm(|a|,|b|)\). Because both \(|a|\) and \(|b|\) are divisors of 12, \(|\langle a,b \rangle| = lcm(|a|,|b|)\) is a divisor of 12. Note that \(12((\mathbb{Z}_4 \oplus \mathbb{Z}_{12})) = 12(a,b) + \mathbb{Z}_4 \oplus \mathbb{Z}_{12} = \mathbb{Z}_4 \oplus \mathbb{Z}_{12}\). So \(|\langle a,b \rangle + \mathbb{Z}_4 \oplus \mathbb{Z}_{12}|\) is a divisor of 12 and there is no order 8 element. Therefore given group is not isomorphic to \(\mathbb{Z}_8\).

So it is isomorphic to \(\mathbb{Z}_4 \oplus \mathbb{Z}_2\).

32. Prove that \(D_4\) cannot be expressed as an internal direct product of two proper subgroups.

If \(D_4 = H \times K\), then because \(|D_4| = 8\), \(|H| = 4\) and \(|K| = 2\) (or vice versa.). Then \(K \approx \mathbb{Z}_2\), and \(H \approx \mathbb{Z}_4\) or \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\). In particular, both \(K\) and \(H\) are Abelian groups.

Because 
\\
\[D_4 = H \times K \approx H \oplus K,\]
\\
\(D_4\) must be Abelian, too. But \(D_4\) is not, so it is not an internal direct product.

34. In \(\mathbb{Z}\), let \(H = \langle 5 \rangle\) and \(K = \langle 7 \rangle\). Prove that \(\mathbb{Z} = HK\). Does \(\mathbb{Z} = \mathbb{Z} \times \mathbb{Z}\)?

Note that \(\mathbb{Z}\) is an additive group. So \(HK = \{a+b \mid a \in H, b \in K\}\).

Because \(gcd(5,7) = 1\), there are two integers \(x\) and \(y\) such that \(5x + 7y = 1\). So for any \(m \in \mathbb{Z}\), \(m = 5mx + 7my \in \langle 5 \rangle \langle 7 \rangle = HK\). Therefore \(\mathbb{Z} = HK\).

But \(H \cap K = \langle lcm(5,7) \rangle = \langle 35 \rangle\). So \(\mathbb{Z} \neq H \times K\).

39. If \(H\) is a normal subgroup of a group \(G\), prove that \(C(H)\), the centralizer of \(H\) in \(G\), is a normal subgroup of \(G\).

We will show that \(x C(H) x^{-1} \subset C(H)\) for all \(x \in G\). Let \(a \in x C(H) x^{-1}\). Then \(a = xyx^{-1}\) for some \(y \in C(H)\). We need to show that \(ah = ha\), or \(aha^{-1} = h\) for all \(h \in H\).

\[aha^{-1} = (xyx^{-1})h(xy^{-1})^{-1} = xy^{-1}hxy^{-1}x^{-1}\]

Because \(H \triangleleft G\), \(x^{-1}hx \in x^{-1}Hx \subset H\). So \(yx^{-1}hx = x^{-1}hxy\). Thus we have \(xyx^{-1}hxy^{-1}x = xx^{-1}hxyy^{-1}x^{-1} = h\). Therefore \(aha^{-1} = h\) and \(a \in C(H)\).
44. Observe from the table for $A_4$ given in Table 5.1 on page 111 that the subgroup given in Example 9 of this chapter is the only subgroup of $A_4$ of order 4. Why does this imply that this subgroup must be normal in $A_4$? Generalize this to arbitrary finite groups.

Generally, if a finite group $G$ has only one subgroup of $H$ of fixed order $k$, then $H \triangleleft G$.

Indeed, for any $x \in G$, $xHx^{-1} \leq G$ and $|xHx^{-1}| = |H| = k$. From the assumption, $xHx^{-1} = H$. This implies that $H \triangleleft G$.

51. Let $N$ be a normal subgroup of $G$ and let $H$ be a subgroup of $G$. If $N$ is a subgroup of $H$, prove that $H/N$ is a normal subgroup of $G/N$ if and only if $H$ is a normal subgroup of $G$.

Suppose that $H$ is a normal subgroup of $G$. Then for any $a \in G$, $aHa^{-1} \subset H$. So for any $hN \in H/N$, $aN \cdot hN \cdot (aN)^{-1} = aha^{-1}N \in H/N$. In other words, $aN(H/N)(aN)^{-1} \subset H/N$ for any $aN \in G/N$. Therefore $H/N \triangleleft G/N$.

Conversely, suppose that $H/N \triangleleft G/N$. Then for any $aN \in G/N$, $aN(H/N)(aN)^{-1} \subset H/N$. So $aha^{-1}N = aN \cdot hN \cdot (aN)^{-1} \in H/N$ for all $h \in H$. Therefore there exists $h' \in H$ such that $aha^{-1}N = h'N$. So $aha^{-1} = h'n$ for some $n \in N$. Then $aha^{-1} = h'n \in H$, because $N \leq H$. This implies that $aHa^{-1} \subset H$. So $H \triangleleft G$.

58. If $N$ and $M$ are normal subgroups of $G$, prove that $NM$ is also a normal subgroup of $G$.

In general, $NM$ is not a subgroup! But if (at least one of) $N$ and $M$ are normal, we can prove that $NM$ is also a subgroup. $e = ee \in NM$, so $NM$ is nonempty. Take $n_1m_1, n_2m_2 \in NM$. Then $(n_1m_1)(n_2m_2)^{-1} = n_1m_1m_2^{-1}n_2^{-1}$. Because $M$ is normal, there is $m_3 \in M$ such that $m_1m_2^{-1}n_2^{-1} = n_2^{-1}m_3$. So $n_1m_1m_2^{-1}n_2^{-1} = n_1n_2^{-1}m_3 \in NM$ and $NM \leq G$.

Because $N$ and $M$ are normal subgroups, $aNa^{-1} \subset N$ and $aMa^{-1} \subset M$ for any $a \in G$. Now $aNa^{-1} = aNa^{-1}aMa^{-1} \subset NM$. Therefore $NM \triangleleft G$ also.