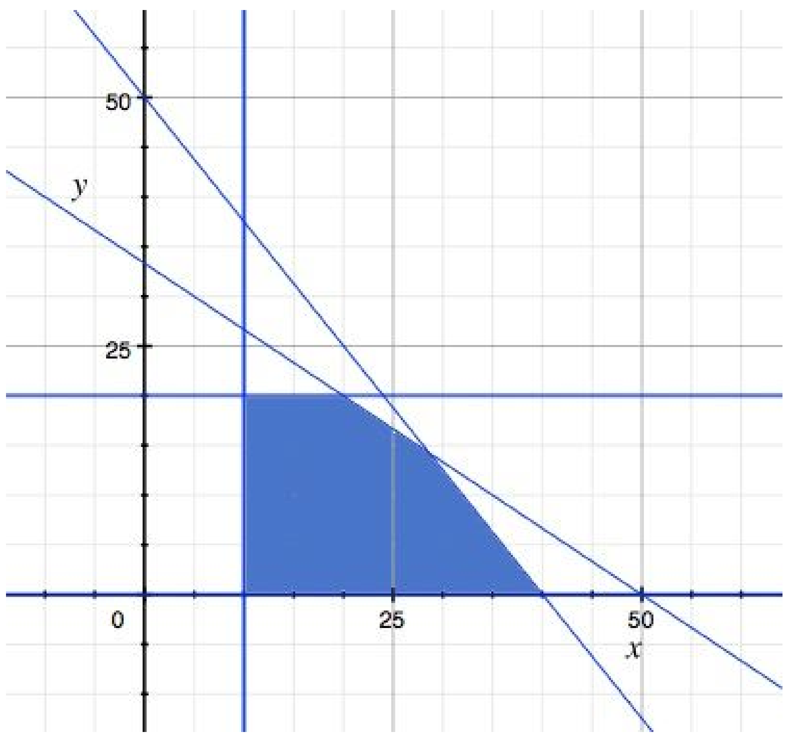
Homework 6 Solution

Section 3.2 $\sim$ 3.3

3.2.10. Use graphical methods to solve the linear programming problem maximizing $z = 10x + 8y$ subject to

$$\begin{aligned} 2x + 3y &\leq 100 \\ 5x + 4y &\leq 200 \\ x &\geq 10 \\ 0 \leq y &\leq 20. \end{aligned}$$



Corner points: $(10, 0), (40, 0), (10, 20)$

$$y = 20, 2x + 3y = 100 \Rightarrow 2x + 60 = 100 \Rightarrow 2x = 100 - 60 = 40 \Rightarrow x = 20$$

$$\Rightarrow (x, y) = (20, 20)$$

$$2x + 3y = 100, 5x + 4y = 200 \Rightarrow 8x + 12y = 400, 15x + 12y = 600$$

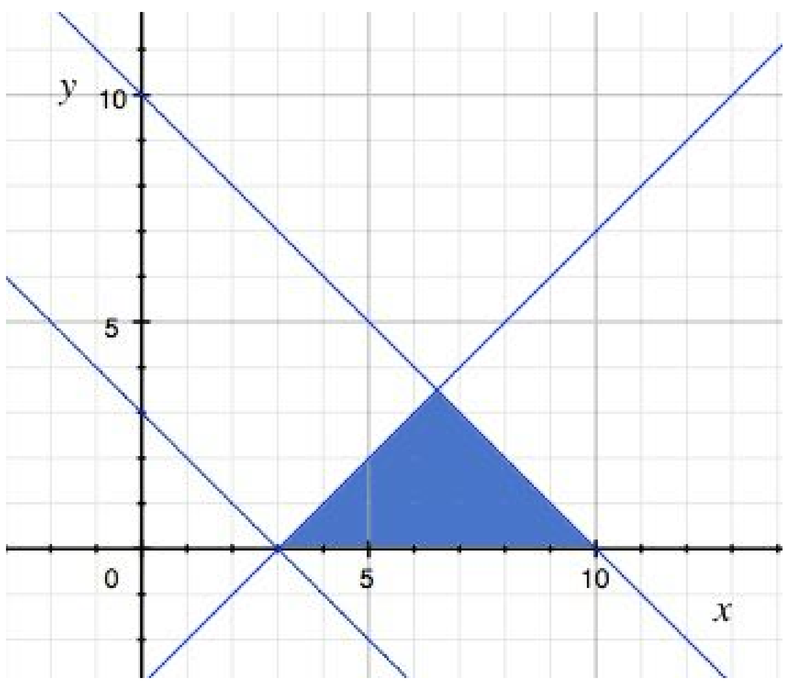
$$\Rightarrow 7x = 200 \Rightarrow x = \frac{200}{7} \Rightarrow 2 \cdot \frac{200}{7} + 3y = 100 \Rightarrow 3y = 100 - \frac{400}{7} = \frac{300}{7}$$
$$\Rightarrow y = \frac{100}{7} \Rightarrow (x, y) = (\frac{200}{7}, \frac{100}{7})$$

corner point	objective function ($z = 10x + 8y$)
$(10, 0)$	$10 \cdot 10 + 8 \cdot 0 = 100$
$(40, 0)$	$10 \cdot 40 + 8 \cdot 0 = 400$
$(10, 20)$	$10 \cdot 10 + 8 \cdot 20 = 260$
$(20, 20)$	$10 \cdot 20 + 8 \cdot 20 = 360$
$(\frac{200}{7}, \frac{100}{7})$	$10 \cdot \frac{200}{7} + 8 \cdot \frac{100}{7} = 400$

On $(40, 0)$, $(\frac{200}{7}, \frac{100}{7})$, and every point between them the maximum occurs and the maximum is 400.

3.2.14. Use graphical methods to solve the linear programming problem maximizing $z = 4x + 6y$ subject to

$$\begin{aligned} 3 \le x + y &\le 10 \\ x - y &\ge 3 \\ x &\ge 0 \\ y &\ge 0. \end{aligned}$$



Corner points: $(3, 0), (10, 0)$

$$x + y = 10, x - y = 3 \Rightarrow 2x = 13 \Rightarrow x = \frac{13}{2}$$

$$\Rightarrow \frac{13}{2} + y = 10 \Rightarrow y = \frac{7}{2} \Rightarrow (x, y) = (\frac{13}{2}, \frac{7}{2})$$

corner point	objective function ($z = 4x + 6y$)
$(3, 0)$	$4 \cdot 3 + 6 \cdot 0 = 12$
$(10, 0)$	$4 \cdot 10 + 6 \cdot 0 = 40$
$(\frac{13}{2}, \frac{7}{2})$	$4 \cdot \frac{13}{2} + 6 \cdot \frac{7}{2} = 47$

On $(\frac{13}{2}, \frac{7}{2})$ the maximum occurs and the maximum is 47.

3.3.8. A manufacturer of refrigerators must ship at least 100 refrigerators to its two West Coast warehouses. Each warehouse holds a maximum of 100 refrigerators. Warehouse A holds 25 refrigerators already, and warehouse B has 20 on hand. It costs \$12 to ship a refrigerator to warehouse A and \$10 to ship one to warehouse B. Union rules require that at least 300 workers be hired. Shipping a refrigerator to warehouse A requires 4 workers, while shipping a refrigerator to warehouse B requires 2 workers. How many refrigerators should be shipped to each warehouse to minimize costs? What is the minimum cost?

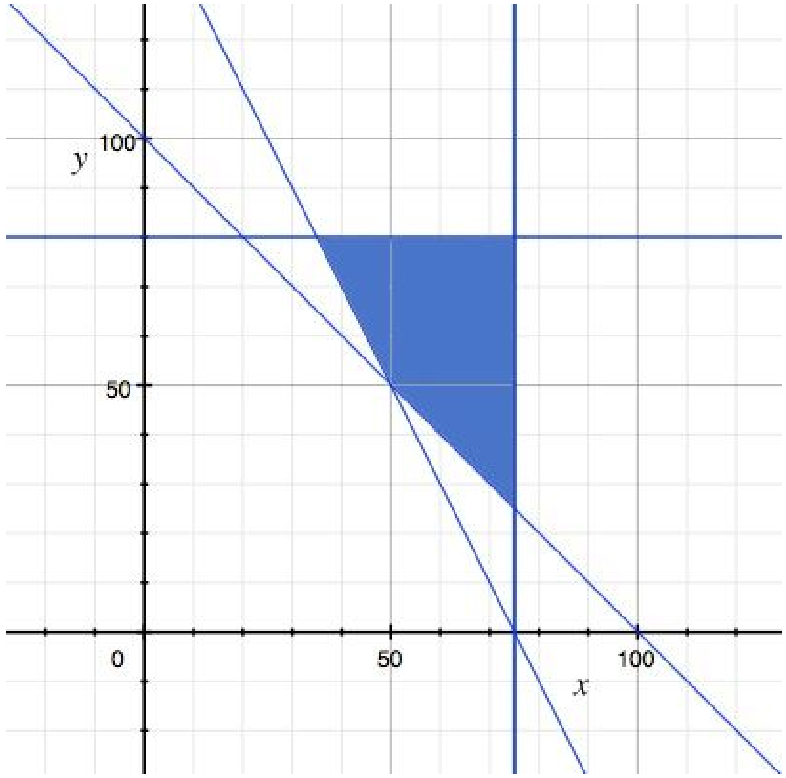
x: the number of refrigerators shipped to warehouse A
y: the number of refrigerators shipped to warehouse B

From the total number of refrigerators, we have $x + y \geq 100$. And because warehouse A already holds 25, $x \leq 100 - 25 = 75$. By the same reason, $y \leq 100 - 20 = 80$. To ship x refrigerators to A and y refrigerators to B, we need $4x + 2y$ workers. So we have another constraint $4x+2y \geq 300$. Finally, the total cost is $z = 12x+10y$.

Therefore we have the following constraints

$$\begin{aligned} x + y &\geq 100 \\ x &\leq 75 \\ y &\leq 80 \\ 4x + 2y &\geq 300 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

with the objective function $z = 12x + 10y$.



Corner points: $(75, 80)$

$$x + y = 100, x = 75 \Rightarrow 75 + y = 100 \Rightarrow y = 25 \Rightarrow (x, y) = (75, 25)$$

$$4x + 2y = 300, y = 80 \Rightarrow 4x + 160 = 300 \Rightarrow x = \frac{300 - 160}{4} = 35$$
$$\Rightarrow (x, y) = (35, 80)$$

$$4x + 2y = 300, x + y = 100 \Rightarrow y = 100 - x \Rightarrow 4x + 2(100 - x) = 300$$
$$\Rightarrow 2x + 200 = 300 \Rightarrow x = 50 \Rightarrow y = 100 - 50 \Rightarrow (x, y) = (50, 50)$$

corner point	objective function ($z = 12x + 10y$)
$(75, 80)$	$12 \cdot 75 + 10 \cdot 80 = 1700$
$(75, 25)$	$12 \cdot 75 + 10 \cdot 25 = 1150$
$(35, 80)$	$12 \cdot 35 + 10 \cdot 80 = 1220$
$(50, 50)$	$12 \cdot 50 + 10 \cdot 50 = 1100$

The minimum occurs and the maximum is 1100 and 50 refrigerators should be shipped to warehouse A and the other 50 should be shipped to warehouse B.

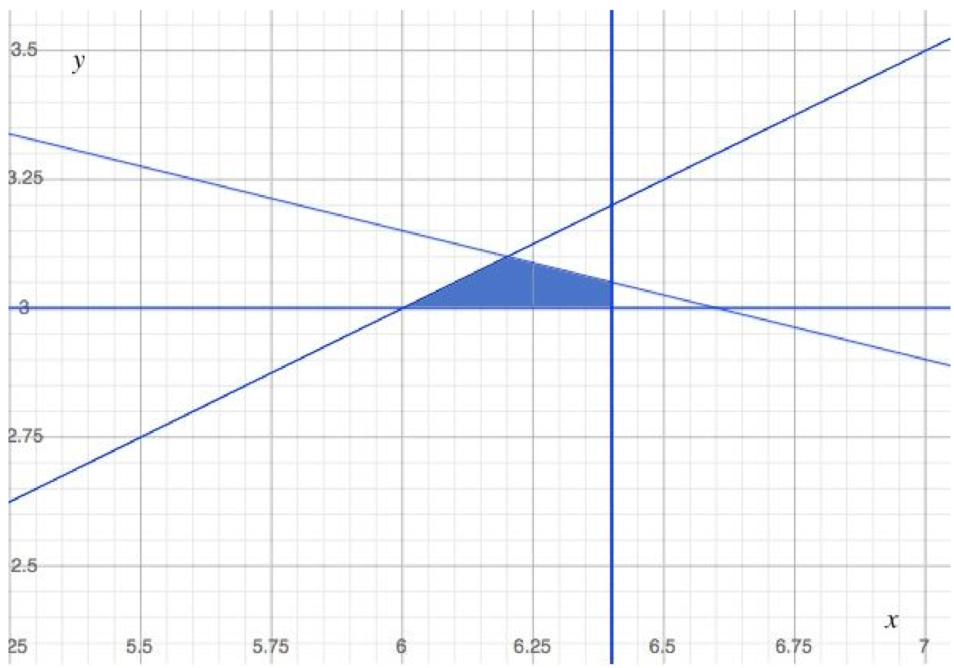
3.3.12. The manufacturing process requires that oil refineries must manufacture at least 2 gal of gasoline for every gallon of fuel oil. To meet the winter demand for fuel oil, at least 3 million gal a day must be produced. The demand for gasoline is no more than 6.4 million gal per day. It takes 15 minutes to ship each million gal of gasoline and 1 hour to ship each million gal of fuel oil out of the warehouse. No more than 4 hours and 39 minutes are available for shipping.

(a) If the refinery sells gasoline for \$2.50 per gal and fuel oil for \$2 per gal, how much of each should be produced to maximize revenue?

Suppose that the refinery produces x million gallons of gas and y million gallons of fuel oil per day. From the first conditions, we can find several constraints:

$$\begin{aligned} x &\geq 2y \\ y &\geq 3 \\ x &\leq 6.4 \\ 15x + 60y &\leq 279 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

The fourth inequality came from the total shipping time (in minutes). The objective function is $z = 2.5x + 2y$.



Corner points: $(6.4, 3)$

$$x = 2y, y = 3 \Rightarrow x = 6 \Rightarrow (x, y) = (6, 3)$$

$$15x + 60y = 279, x = 6.4 \Rightarrow 96 + 60y = 279 \Rightarrow y = \frac{279 - 96}{60} = 3.05$$
$$\Rightarrow (x, y) = (6.4, 3.05)$$
$$15x + 60y = 279, x = 2y \Rightarrow y = \frac{1}{2}x \Rightarrow 15x + 60 \cdot \frac{1}{2}x = 279$$
$$\Rightarrow 45x = 279 \Rightarrow x = \frac{31}{5} = 6.2$$
$$\Rightarrow y = \frac{1}{2} \cdot 6.2 = 3.1 \Rightarrow (x, y) = (6.2, 3.1)$$

corner point	objective function ($z = 2.5x + 2y$)
$(6.4, 3)$	$2.5 \cdot 6.4 + 2 \cdot 3 = 22$
$(6, 3)$	$2.5 \cdot 6 + 2 \cdot 3 = 21$
$(6.4, 3.05)$	$2.5 \cdot 6.4 + 2 \cdot 3.05 = 22.1$
$(6.2, 3.1)$	$2.5 \cdot 6.2 + 2 \cdot 3.1 = 21.7$

The maximum occurs when 6.4 million gallons of gasoline and 3.05 million gallons of fuel oil is produced.

(b) Find the maximum revenue.

22.1 million dollars (be careful about the unit!)

(c) Suppose the price for fuel oil begins to increase. Beyond what amount would this price have to increase before a different amount of gasoline and fuel oil should be produced to maximize revenue?

No matter what new price is, it is clear that at one of two corner points $(6.4, 3.05)$ and $(5.32, 3.32)$ the maximum occurs. Suppose that p is new price of fuel oil. Then the new objective function is $z = 2.5x + py$. With this objective function, at two corner points

corner point	objective function ($z = 2.5x + py$)
$(6.4, 3.05)$	$2.5 \cdot 6.4 + p \cdot 3.05 = 3.05p + 16$
$(6.2, 3.1)$	$2.5 \cdot 6.2 + p \cdot 3.1 = 3.1p + 15.5$

$$3.05p + 16 = 3.1p + 15.5 \Rightarrow 3.1p - 3.05p = 16 - 15.5$$
$$\Rightarrow 0.05p = 0.5 \Rightarrow p = 10$$

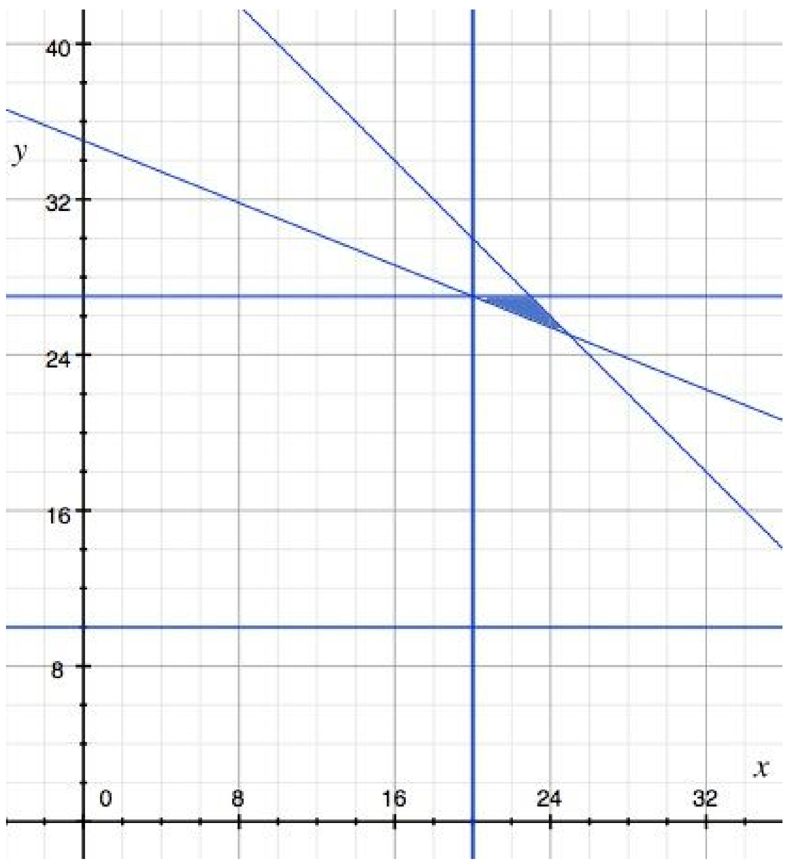
Therefore beyond \$10, at a different amount of gasoline and fuel oil the maximum revenue occurs.

3.3.14. A small country can grow only two crops for export, coffee and cocoa. The country has 500,000 hectares of land available for the crops. Long-term contracts require that at least 100,000 hectares be devoted to coffee and at least 200,000 hectares to cocoa. Cocoa must be processed locally, and production bottlenecks limit cocoa to 270,000 hectares. Coffee requires two workers per hectare, with cocoa requiring five. No more than 1,750,000 people are available for working with these crops. Coffee produces a profit of \$220 per hectare and cocoa a profit of \$550 per hectare. How many hectares should the country devote to each crop in order to maximize profit? Find the maximum profit.

x: the area (ten thousand hectares) for growing coffee
y: the area (ten thousand hectares) for growing cocoa

$$\begin{aligned} x + y &\leq 50 \\ x &\geq 20 \\ y &\geq 10 \\ y &\leq 27 \\ 2x + 5y &\leq 175 \end{aligned}$$

The profit is $z = 220x + 550y$.



Corner points: $(20, 27)$

$$x + y = 50, y = 27 \Rightarrow x = 50 - 27 = 23 \Rightarrow (x, y) = (23, 27)$$

$$2x+5y = 175, x+y = 50 \Rightarrow y = 50-x \Rightarrow 2x+5(50-x) = 175 \Rightarrow -3x+250 = 175$$

$$\Rightarrow x = 25 \Rightarrow y = 50 - 25 = 25 \Rightarrow (x, y) = (25, 25)$$

corner point	objective function ($z = 220x + 550y$)
$(20, 27)$	$220 \cdot 20 + 550 \cdot 27 = 19250$
$(23, 27)$	$220 \cdot 23 + 550 \cdot 27 = 19910$
$(25, 25)$	$220 \cdot 25 + 550 \cdot 25 = 19250$

The maximum occurs when they grow 230,000 hectares of coffee and 270,000 hectares of cocoa and the maximum profit is $199,100,000.

3.3.16. A flash drive manufacturer has 370 boxes of a particular drive in warehouse I and 290 boxes of the same drive in warehouse II. A computer store in San Jose orders 350 boxes of the drive, and another store in Memphis orders 300 boxes. The shipping costs per box to these stores from the two warehouses are shown in the following table.

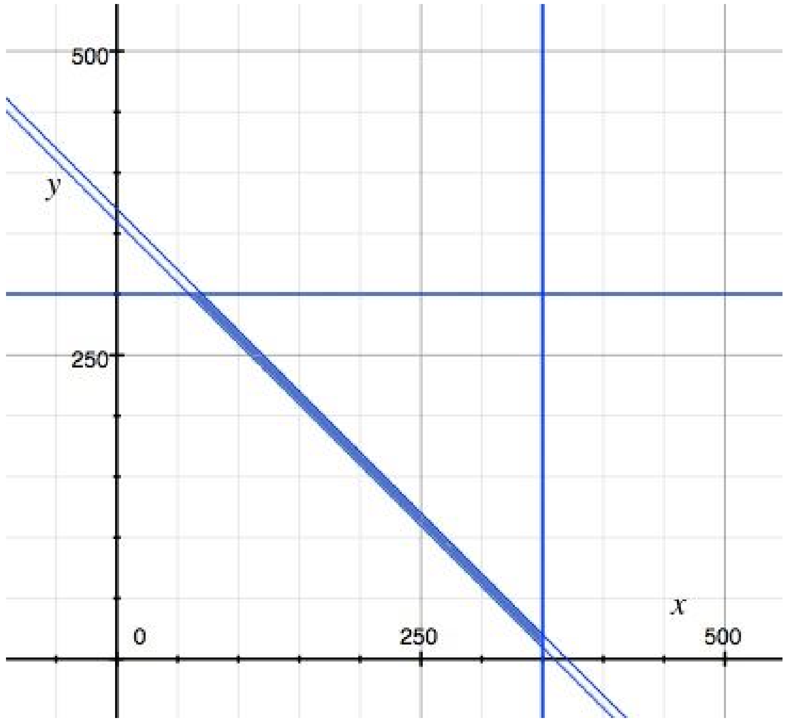
	San Jose	Memphis
Warehouse I	$2.50	$2.20
Warehouse II	$2.30	$2.10

How many boxes should be shipped to each city from each warehouse to minimize shipping costs? What is the minimum cost?

x: number of boxes from warehouse I to San Jose
y: number of boxes from warehouse I to Memphis

Then the number of boxes from warehouse II to San Jose is $350 - x$, and the number of boxes from warehouse II to Memphis is $300 - y$. Then $x + y \leq 370$ and $(350 - x) + (300 - y) \leq 290$. The second inequality is same with $x + y \geq 360$. Also, we have $0 \leq x \leq 350$, $0 \leq y \leq 300$. The objective function, which is the profit, is

$$z = 2.5x + 2.2y + 2.3(350 - x) + 2.1(300 - y) = 0.2x + 0.1y + 1435$$



Corner points:

$$x + y = 370, y = 300 \Rightarrow x = 70 \Rightarrow (x, y) = (70, 300)$$

$$x + y = 360, y = 300 \Rightarrow x = 60 \Rightarrow (x, y) = (60, 300)$$

$$x + y = 370, x = 350 \Rightarrow y = 20 \Rightarrow (x, y) = (350, 20)$$

$$x + y = 360, x = 350 \Rightarrow y = 10 \Rightarrow (x, y) = (350, 10)$$

corner point	objective function ($z = 0.2x + 0.1y + 1435$)
$(70, 300)$	$0.2 \cdot 70 + 0.1 \cdot 300 + 1435 = 1479$
$(60, 300)$	$0.2 \cdot 60 + 0.1 \cdot 300 + 1435 = 1477$
$(350, 20)$	$0.2 \cdot 350 + 0.1 \cdot 20 + 1435 = 1507$
$(350, 10)$	$0.2 \cdot 350 + 0.1 \cdot 10 + 1435 = 1506$

The minimum cost is $1477 and it occurs when 60 boxes are shipped from warehouse I to San Jose, 300 boxes from warehouse I to Memphis, 290 boxes from warehouse II to San Jose.