Homework 4 Solution

Section 2.2 $\sim$ 2.3.

- Do not abbreviate your answer. Write everything in full sentences.
- Write your answer neatly. If I couldn't understand it, you'll get 0 point.
- You may discuss with your classmates. But do not copy directly.

1. Let A, B, and C be three sets. Suppose that $A \subset B$ and $B \subset C$. Show that $A \subset C$.

   Let $x \in A$. Because $A \subset B$, $x \in B$. Since $B \subset C$, $x \in C$. Therefore $A \subset C$.

2. Let $W = \{n \in \mathbb{Z} \mid n = x - y \text{ for some } x, y \in \mathbb{N}\}$. (Note that in our definition of natural numbers, 0 is excluded.) Show that $W = \mathbb{Z}$.

   Let $n \in W$. Then by definition, $n \in \mathbb{Z}$, so $W \subset \mathbb{Z}$. Conversely, let $n \in \mathbb{Z}$. If $n > 0$, then $n = (n+1) - 1$ and both $n+1$ and 1 are natural numbers. If $n = 0$, then $n = 1 - 1$. If $n < 0$, then $n = 1 - (-n+1)$ and both 1 and $-n+1$ are natural numbers. Therefore in any case, n can be written as a difference of two natural numbers. So $n \in W$. Thus $\mathbb{Z} \subset W$. This proves $W = \mathbb{Z}$.

3. Consider a collection X defined by

   $$X = \{y \mid y \notin y\}.$$

   Show that it makes a contradiction regardless you assume $X \in X$ or $X \notin X$. (This is a famous example so-called *Russell's paradox*, which is a collection we want to exclude from the definition of sets.)

   If $X \in X$, then by definition of the set X, $X \notin X$. If $X \notin X$, then by definition of the set X again, X does not have the property $X \notin X$. Therefore $X \in X$. In any case, we have a contradiction.

4. Compute the following sets.

   (a) $\mathcal{P}(\mathcal{P}(\emptyset))$

   $$\mathcal{P}(\emptyset) = \{\emptyset\}$$

   $$\mathcal{P}(\mathcal{P}(\emptyset)) = \{\emptyset, \{\emptyset\}\}$$

   Please distinguish $\emptyset$ and $\{\emptyset\}$.

   (b) For $A = \{1, 2, 3\}$, $A \times \emptyset$.

   By definition, $A \times \emptyset = \{(x, y) \mid x \in A, y \in \emptyset\}$. But there is no $y \in \emptyset$. Therefore $A \times \emptyset = \emptyset$.

(c) For $A = \{a, b, c, d, e\}$ and $B = \{b, d, f\}$, $(A \cap B) \times (A \setminus B)$.

$$A \cap B = \{b, d\}, A \setminus B = \{a, c, e\}$$

$$(A \cap B) \times (A \setminus B) = \{(b, a), (b, c), (b, e), (d, a), (d, c), (d, e)\}$$

5. Determine whether the given statement is true or false, and explain the reason. (You don't need to give a formal proof.)

   (a) $\emptyset \subset \emptyset$.
   True. $\emptyset$ is a subset of every set, including $\emptyset$.

   (b) $\emptyset \in \emptyset$.
   False. $\emptyset$ does not have any element, by definition.

   (c) For any nonempty set A, $A \subset A \times A$.
   False. Because every element of $A \times A$ is of the form (x, y) where $x, y \in A$, none of $x \in A$ is in $A \times A$.

   (d) For any nonempty set A, $A \subset \mathcal{P}(A)$.
   False. For any $x \in A$, x is not a subset of A, so $x \notin \mathcal{P}(A)$. (But $\{x\} \in \mathcal{P}(A)$.)

   (e) For any nonempty set A, $A \times A \subset \mathcal{P}(A)$.
   False. An element of $A \times A$ is (x, y) for some $x, y \in A$. But (x, y) is different from $\{x, y\}$, which is an element of $\mathcal{P}(A)$. Since (x, y) is not a subset, $(x, y) \notin \mathcal{P}(A)$.

6. Let $f : \mathbb{N} \to \mathbb{Z}$ be a function defined by

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ \frac{1-n}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

   (a) Show that f is one-to-one correspondence.
   Let $a, b \in \mathbb{N}$. Suppose that $f(a) = f(b)$. If $f(a) > 0$, then both $f(a)$ and $f(b)$ are positive, so $f(a) = \frac{a}{2}$ and $f(b) = \frac{b}{2}$. Then $\frac{a}{2} = f(a) = f(b) = \frac{b}{2}$. Therefore $a = b$. If $f(a) \le 0$, then both $f(a)$ and $f(b)$ are non positive. So $f(a) = \frac{1-a}{2}$ and $f(b) = \frac{1-b}{2}$. So we have $\frac{1-a}{2} = f(a) = f(b) = \frac{1-b}{2}$. This implies $1 - a = 1 - b$, and $a = b$. So in any case, we have $a = b$ and f is injective.
   Let $c \in \mathbb{Z}$. If $c > 0$, then $2c \in \mathbb{N}$ and $f(2c) = \frac{2c}{2} = c$. If $c \le 0$, $-2c \ge 0$, so $1 - 2c \in \mathbb{N}$. Moreover, $f(1 - 2c) = \frac{1-(1-2c)}{2} = \frac{2c}{2} = c$. Therefore in any case, we can find $d \in \mathbb{N}$ such that $f(d) = c$. Therefore f is onto. So we can conclude that f is one-to-one correspondence.
   Wait, this result is very surprising. $\mathbb{N}$ is contained in $\mathbb{Z}$ and there are infinitely many elements of $\mathbb{Z} \setminus \mathbb{N}$, but we could make a bijective function between $\mathbb{N}$ and $\mathbb{Z}$. This is a big difference between finite sets and infinite sets.

(b) Find a formula for f^{-1}.

The proof in (a) also provides the formula of f^{-1}. Let $x \in \mathbb{Z}$. If $x > 0$, then $f(2x) = x$. So $f^{-1}(x) = 2x$. If $x \le 0$, $f(1-2x) = x$. Therefore $f^{-1}(x) = 1-2x$. In summary,

$$f^{-1}(x) = \begin{cases} 2x, & x > 0 \\ 1-2x, & x \le 0. \end{cases}$$

7. Let $f : X \to Y$, $g : Y \to Z$ be two functions.

(a) Suppose that f, g are injective. Prove that $g \circ f$ is injective.

Let $a, b \in X$. Suppose that $(g \circ f)(a) = (g \circ f)(b)$. Then by definition, $g(f(a)) = g(f(b))$. Because g is injective, this implies $f(a) = f(b)$. Since f is injective, $a = b$. Therefore $g \circ f$ is injective.

(b) Suppose that f, g are surjective. Prove that $g \circ f$ is surjective.

Let $c \in Z$. Because g is surjective, there is $d \in Y$ such that $g(d) = c$. Since f is surjective, there is $e \in X$ such that $f(e) = d$. Then $(g \circ f)(e) = g(f(e)) = g(d) = c$. Therefore $g \circ f$ is surjective.

8. Let $f : X \to Y$ and $g : Y \to Z$ be two functions. Suppose that $g \circ f$ is bijective.

(a) Show that f is injective.

Let $a, b \in X$ and suppose that $f(a) = f(b)$. Then $g(f(a)) = g(f(b))$. By definition of the composition function, $(g \circ f)(a) = (g \circ f)(b)$. Because $g \circ f$ is bijective, it is an injective function. So $a = b$. Therefore f is injective.

(b) Prove that g is surjective.

Let $c \in Z$. Because $g \circ f$ is bijective, $g \circ f$ is surjective. So there is $d \in X$ such that $(g \circ f)(d) = c$. Now $g(f(d)) = (g \circ f)(d) = c$. Therefore g is surjective.

9. Let $f : X \to Y$ and $g : Y \to X$ be two functions. Suppose that $g \circ f = \mathrm{id}_X$. Give an example of f and g which are not bijective. (Therefore $g \circ f = \mathrm{id}_X$ is *not* sufficient to guarantee the existence of f^{-1} or g^{-1}.)

Let $X = \{a, b\}$ and $Y = \{1, 2, 3\}$. Define $f : X \to Y$ as $f(a) = 1$, $f(b) = 2$. And define $g : Y \to X$ as $g(1) = a$, $g(2) = b$, and $g(3) = b$. Then clearly f is injective, g is surjective, and $g \circ f = \mathrm{id}_X$. But f is not surjective because there is no $x \in X$ such that $f(x) = 3$. And g is not injective because $g(2) = b = g(3)$.

10. Let $f : X \to Y$ be a function. And let $A, B \subset X$.

(a) Show that $f(A \cap B) \subset f(A) \cap f(B)$.

Let $a \in f(A \cap B)$. Then there is $b \in A \cap B$ such that $f(b) = a$. Then $b \in A$, so $a = f(b) \in f(A)$. Similarly, because $b \in B$, $a = f(b) \in f(B)$. Therefore $a \in f(A) \cap f(B)$.

(b) Is it true that $f(A \cap B) = f(A) \cap f(B)$? Prove it if it is true, and give a counterexample if it is false.

Let $X = \{a, b, c\}$, $Y = \{1, 2\}$, and let $f : X \to Y$ be a function defined by $f(a) = 1$, $f(b) = 2$, and $f(c) = 1$. Let $A = \{a, b\}$ and $B = \{b, c\}$. Then $f(A) = \{1, 2\}$ and $f(B) = \{1, 2\}$. Therefore $f(A) \cap f(B) = \{1, 2\}$. On the other hand, $A \cap B = \{b\}$. So $f(A \cap B) = \{2\}$. Therefore $f(A \cap B) \neq f(A) \cap f(B)$.

11. Let $f : X \to Y$ be a function. And let $A, B \subset Y$.

(a) Show that $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$.

Let $a \in f^{-1}(A \cap B)$. Then $f(a) \in A \cap B$. So $f(a) \in A$. This implies $a \in f^{-1}(A)$. Similarly, $f(a) \in B$ so $a \in f^{-1}(B)$. Therefore $a \in f^{-1}(A) \cap f^{-1}(B)$. Therefore $f^{-1}(A \cap B) \subset f^{-1}(A) \cap f^{-1}(B)$.

(b) Is it true that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$? Prove it if it is true, and give a counterexample if it is false.

Let $a \in f^{-1}(A) \cap f^{-1}(B)$. Then $a \in f^{-1}(A)$. So $f(a) \in A$. Similary $a \in f^{-1}(B)$. Thus $f(a) \in B$. Therefore $f(a) \in A \cap B$. This implies $a \in f^{-1}(A \cap B)$. So we have $f^{-1}(A) \cap f^{-1}(B) \subset f^{-1}(A \cap B)$. The opposite inclusion was shown in (a). Therefore $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

12. Let $f : X \to Y$ be a function. Let $B \subset Y$.

(a) Prove that $f(f^{-1}(B)) \subset B$.

Let $a \in f(f^{-1}(B))$. Then there is $b \in f^{-1}(B)$ such that $f(b) = a$. Because $b \in f^{-1}(B)$, by the definition of the inverse image, $a = f(b) \in B$. Therefore $f(f^{-1}(B)) \subset B$.

(b) Is it true that $f(f^{-1}(B)) = B$? Prove it if it is true, and give a counterexample if it is false.

Let $X = \{a, b\}$, $Y = \{1, 2, 3\}$, and $f : X \to Y$ be a function defined by $f(a) = 1$ and $f(b) = 2$. Let $B = \{2, 3\}$. Then $f^{-1}(B) = \{b\}$. And $f(f^{-1}(B)) = f(\{b\}) = \{2\}$. Therefore $f(f^{-1}(B)) \neq B$.