# MATH 1207 R04 FINAL SOLUTION

SPRING 2017 - MOON

(1) Let $f(x) = x\sin x$.

(a) (4 pts) Find $f'(x)$.

$$f'(x) = \left(\frac{d}{dx}x\right)\sin x + x\left(\frac{d}{dx}\sin x\right) = \sin x + x\cos x$$

- Applying the product rule correctly and getting $\left(\frac{d}{dx}x\right)\sin x + x\left(\frac{d}{dx}\sin x\right)$: 2 pts.
- Getting the answer $\sin x + x\cos x$: 4 pts.

(b) (4 pts) Find $\int f(x)dx$.

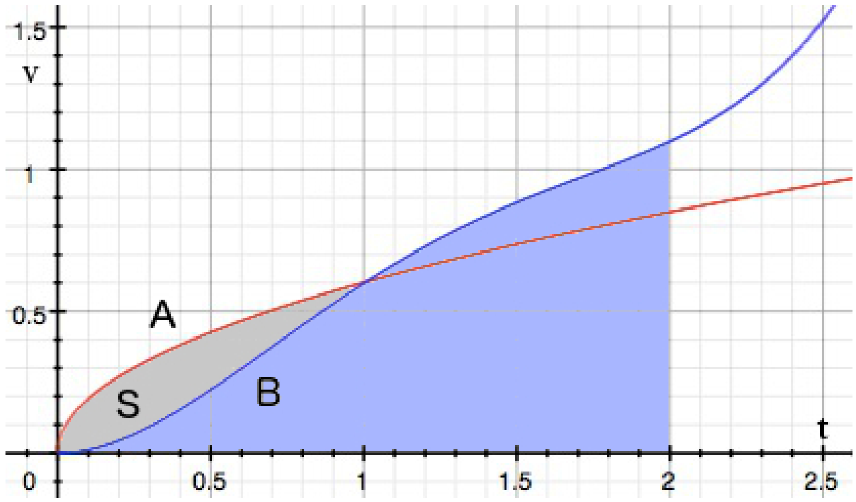
$$u = x, dv = \sin x dx \Rightarrow du = dx, v = -\cos x$$

$$\int x\sin x dx = x(-\cos x) - \int -\cos x dx$$
$$= -x\cos x + \int \cos x dx = -x\cos x + \sin x + C$$

- Finding appropriate $u = x, dv = \sin x dx$: 2 pts.
- Applying integration by parts and getting $x(-\cos x) - \int -\cos x dx$: 3 pts.
- Getting the answer $-x\cos x + \sin x + C$: 4 pts.

*Date*: May 10, 2017.

(2) Two cars $A$ and $B$, start side by side and accelerate from rest. The figure shows the graphs of their *velocity* functions. The unit of time is minute.




(a) (2 pts) On the graph above, indicate the region whose area is the moving distance of $B$ during first 2 minutes.

See the blue region on the graph above.

(b) (2 pts) Which car is ahead after one minute? Explain your answer.

Car $A$ is ahead, because the area under the graph, which is the moving distance is larger.

(c) (2 pts) What is the physical meaning of the area of the shaded region $S$?

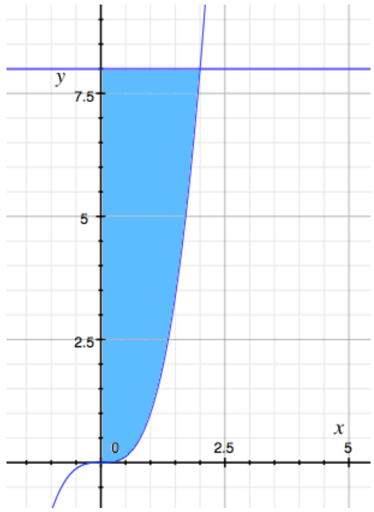
It is the distance between two cars $A$ and $B$ after one minute.

- -1 pt if one does not mention the time.

(d) (2 pts) Estimate the time at which the cars are again side by side. (You don't need to find the precise time. Give an approximation and explain your answer.)

After two minutes, the area under the graph of $A$ and that of $B$ are same. Thus the moving distance of $A$ and $B$ during the first two minutes are same and the cars are side by side.

(3) (5 pts) Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the $x$-axis.




Cross section: washer with the outer radius $8$ and the inner radius $x^3$.

$$\int_0^2 \pi(8^2 - (x^3)^2)dx = \pi \int_0^2 64 - x^6 dx = \pi \left(64x - \frac{x^7}{7}\right)\Bigg]_0^2$$
$$= \pi \left(128 - \frac{128}{7}\right) = \frac{768\pi}{7}$$

- Sketching the planar region: 1 pt.
- Describing the shape of washer: 2 pts.
- Stating the volume formula $\int_0^2 \pi(8^2 - (x^3)^2)dx$: 4 pts.
- Getting the answer $\frac{768\pi}{7}$: 5 pts.

(4) (5 pts) Find the sum of

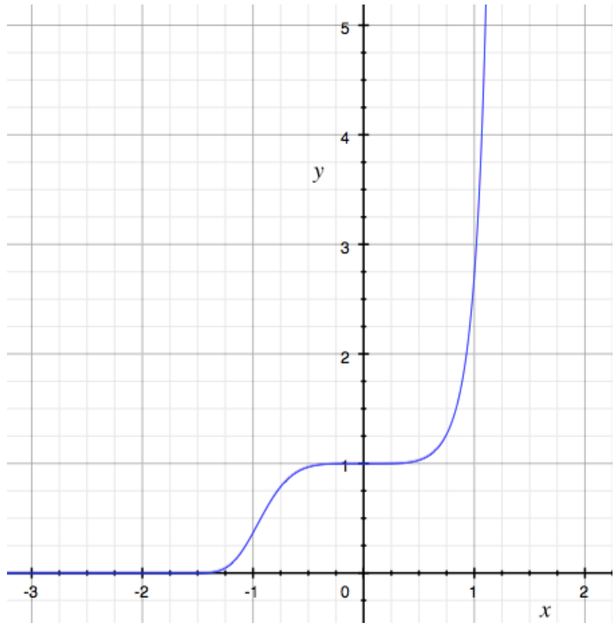
$$\sum_{n=1}^{\infty} 9^{-n+1}4^n.$$

$$\sum_{n=1}^{\infty} 9^{-n+1}4^n = \sum_{n=1}^{\infty} \frac{4^n}{9^{n-1}} = \sum_{n=1}^{\infty} 9 \cdot \frac{4^n}{9^n} = \sum_{n=1}^{\infty} 9 \left(\frac{4}{9}\right)^n$$
$$= \sum_{n=0}^{\infty} 9 \left(\frac{4}{9}\right)^n - 9 = \frac{9}{1 - \frac{4}{9}} - 9 = \frac{81}{5} - 9 = \frac{36}{5}$$

- Describing the given series as a geometric series: 2 pts.
- By using the summation formula, getting $\frac{9}{1 - \frac{4}{9}} - 9$: 4 pts.
- Getting the answer $\frac{36}{5}$: 5 pts.
- If one compute the sum $\sum_{n=0}^{\infty} 9^{-n+1}4^n$: -2 pts.

(5) Let $f(x) = e^{(x^5)}$.

(a) (2 pts) Find the domain and the range of $f$. You don't need to explain the reason.




Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

- 1 pt each.

(b) (2 pts) Show that $f$ is one-to-one.

$f'(x) = e^{x^5} 5x^4 \geq 0$ and $f'(x) > 0$ except $x = 0$. Therefore $f$ is an increasing function. So it is one-to-one.

(c) (2 pts) Find $f^{-1}(e)$.

$$f^{-1}(e) = x \Leftrightarrow f(x) = e \Leftrightarrow e^{x^5} = e \Leftrightarrow x = 1$$

$f^{-1}(e) = 1$.

(d) (2 pts) By using *inverse function theorem*, find $(f^{-1})'(e)$.

$$(f^{-1})'(e) = \frac{1}{f'(1)} = \frac{1}{e^{1^5} 5 \cdot 1^4} = \frac{1}{5e}$$

- Stating inverse function theorem correctly and getting $(f^{-1})'(e) = \dfrac{1}{f'(1)}$: 1 pt.
- Getting the answer $\dfrac{1}{5e}$: 2 pts.

(6) A common inhabitant of human intestines is the bacterium *Escherichia coli*, named after the German pediatrician Theodor Escherich, who identified it in 1885. A cell of this bacterium in a nutrient-broth medium divides into two cells every 20 minutes. The initial population of a culture is 50 cells.

(a) (5 pts) Find an expression for the number of cells after $t$ hours.

Let $C(t)$ be the number of cells after $t$ hours. Then $C(t) = C_0 e^{kt}$ for some constants $C_0$ and $k$.

$$50 = C(0) = C_0 e^0 \Rightarrow C_0 = 50 \Rightarrow C(t) = 50e^{kt}$$

$$100 = C(\frac{1}{3}) = 50e^{\frac{k}{3}} \Rightarrow e^{\frac{k}{3}} = \frac{100}{50} = 2 \Rightarrow \frac{k}{3} = \ln 2 \Rightarrow k = 3\ln 2$$
$$C(t) = 50e^{(3\ln 2)t}$$

- Writing a general form $C(t) = C_0 e^{kt}$: 2 pts.
- Finding $C_0 = 50$: 3 pts.
- Getting the answer $C(t) = 50e^{(3\ln 2)t}$: 5 pts.

(b) (3 pts) When will the population reach a million cells?

$$1000000 = C(t) = 50e^{(3\ln 2)t} \Rightarrow e^{(3\ln 2)t} = \frac{1000000}{50} = 20000 \Rightarrow (3\ln 2)t = \ln 20000$$

$$t = \frac{\ln 20000}{3\ln 2} \approx 4.7625 \text{ hours}$$

- Setting up the equation $1000000 = C(t) = 50e^{(3\ln 2)t}$: 1 pt.
- Getting the answer 4.7625 hours: 3 pts.
- Writing the answer without using the unit: -1 pt.

(7) (a) (6 pts) Evaluate the integral

$$\int_3^\infty \frac{1}{x(\ln x)^4}dx.$$

$$\int_3^\infty \frac{1}{x(\ln x)^4}dx = \lim_{t\to\infty}\int_3^t \frac{1}{x(\ln x)^4}dx$$

$$\int \frac{1}{x(\ln x)^4}dx \overset{u=\ln x, du=\frac{1}{x}dx}{=} \int \frac{1}{u^4}du = \int u^{-4}du = -\frac{1}{3}u^{-3}+C = -\frac{1}{3u^3}+C = -\frac{1}{3(\ln x)^3}+C$$

$$\int_3^t \frac{1}{x(\ln x)^4}dx = -\frac{1}{3(\ln x)^3}\bigg]_3^t = \frac{1}{3(\ln 3)^3} - \frac{1}{3(\ln t)^3}$$

$$\lim_{t\to\infty}\int_3^t \frac{1}{x(\ln x)^4}dx = \lim_{t\to\infty}\frac{1}{3(\ln 3)^3} - \frac{1}{3(\ln t)^3} = \frac{1}{3(\ln 3)^3}$$

- Stating the definition $\lim_{t\to\infty}\int_3^t \frac{1}{x(\ln x)^4}dx$ of the improper integral: 2 pts.
- By using substitution, evaluate the integral $\int_3^t \frac{1}{x(\ln x)^4}dx = \frac{1}{3(\ln 3)^3} - \frac{1}{3(\ln t)^3}$: 5 pts.
- Evaluating the limit and getting $\frac{1}{3(\ln 3)^3}$: 6 pts.

(b) (2 pts) Determine the convergence or divergence of

$$\sum_{n=3}^\infty \frac{1}{n(\ln n)^4}.$$

Because $f(x) = \frac{1}{x(\ln x)^4}$ is clearly positive and decreasing, we can apply the integral test.

The convergence of the integral $\int_3^\infty \frac{1}{x(\ln x)^4}dx$ implies the convergenceof the series.

- Mentioning the integral test: 1 pt.
- Getting the convergence: 2 pts.

(8) Determine whether the given series is convergent or divergent.

(a) (5 pts)

$$\sum_{n=1}^{\infty} \frac{n^3+7n+4}{2n^6-3n^4+2}$$

$$\lim_{n\to\infty} \frac{\frac{n^3+7n+4}{2n^6-3n^4+2}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^6+7n^4+4n^3}{2n^6-3n^4+2} = \lim_{n\to\infty} \frac{1+\frac{7}{n^2}+\frac{4}{n^3}}{2-\frac{3}{n^2}+\frac{2}{n^6}} = \frac{1}{2} \neq 0$$

Note that $\sum_{n=1}^{\infty} \frac{1}{n^3} < \infty$. By limit comparison test, $\sum_{n=1}^{\infty} \frac{n^3+7n+4}{2n^6-3n^4+2} < \infty$, too.

- Getting the convergence: 1 pt.
- Using an appropriate convergence test: + 4 pts.

(b) (5 pts)

$$\sum_{n=1}^{\infty} \frac{n}{n+1}$$

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$$

Thus $\sum_{n=1}^{\infty} \frac{n}{n+1} = \infty$.

- Getting the divergence: 1 pt.
- Applying an appropriate convergence test correctly: + 4 pts.

(9) (7 pts) Determine the interval of convergence of the power series

$$\sum_{n=3}^{\infty}(-1)^n\frac{\ln n}{n}(5x)^n.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{\ln(n+1)}{n+1}(5x)^{n+1}}{(-1)^n\frac{\ln n}{n}(5x)^n}\right| = \lim_{n\to\infty}\frac{\ln(n+1)n}{(\ln n)(n+1)}|5x|$$

$$= \left(\lim_{n\to\infty}\frac{\ln(n+1)}{\ln n}\right)\left(\lim_{n\to\infty}\frac{n}{n+1}\right)|5x| = |5x|$$

The series is convergent if $|5x| < 1$, or equivalently,

$$-\frac{1}{5} < x < \frac{1}{5}.$$

When $x = \frac{1}{5}$,

$$\sum_{n=3}^{\infty}(-1)^n\frac{\ln n}{n}(5x)^n = \sum_{n=3}^{\infty}(-1)^n\frac{\ln n}{n} < \infty$$

by alternating series test. When $x = -\frac{1}{5}$,

$$\sum_{n=3}^{\infty}(-1)^n\frac{\ln n}{n}(5x)^n = \sum_{n=3}^{\infty}(-1)^n\frac{\ln n}{n}(-1)^n = \sum_{n=3}\frac{\ln n}{n} \ge \sum_{n=3}\frac{1}{n} = \infty.$$

By comparison test, the series is divergent.

Therefore the interval of convergence is

$$\left(-\frac{1}{5}, \frac{1}{5}\right].$$

- Applying the ratio test and computing the limit $\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{\ln(n+1)}{n+1}(5x)^{n+1}}{(-1)^n\frac{\ln n}{n}(5x)^n}\right|$: +2 pts.
- Finding the limit $|5x|$: +2 pts.
- Checking the convergence at two endpoints with appropriate convergence test: +3 pts.
- Checking only one endpoint: +2 pts.

(10) (a) (5 pts) Describe the power series

$$\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

as a rational function (a fraction).

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

Take the derivative:

$$\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$$

- Stating $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$: 2 pts.
- By taking the derivative, getting the rational function $\frac{1}{(1-x)^2}$: 5 pts.

(b) (2 pts) Find the radius of convergence of the series in (a). You don't need to write the reason.

The radius of convergence of $\sum_{n=1}^{\infty} nx^{n-1}$ is equal to that of $\sum_{n=0}^{\infty} x^n$, which is 1.

(c) (5 pts) Find the sum of the infinite series

$$\sum_{n=1}^{\infty} \frac{n^2}{3^n}.$$

$$\sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$$

$$\sum_{n=1}^{\infty} nx^{n} = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \frac{x}{(1-x)^2}$$

$$\sum_{n=1}^{\infty} n^2x^{n-1} = 1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots = \frac{1(1-x)^2 - x \cdot 2(1-x)(-1)}{(1-x)^4} = \frac{1+x}{(1-x)^3}$$

$$\sum_{n=1}^{\infty} n^2x^{n} = 1^2x + 2^2x^2 + 3^2x^3 + 4^2x^4 + \cdots = \frac{x(1+x)}{(1-x)^3}$$

$$\sum_{n=1}^{\infty} \frac{n^2}{3^n} = \sum_{n=1}^{\infty} n^2 \left(\frac{1}{3}\right)^n = \frac{\frac{1}{3}(1+\frac{1}{3})}{(1-\frac{1}{3})^3} = \frac{3}{2}$$

- Each line of the computation: +1 pt.

(11) (a) (5 pts) Let $g(x) = x \ln x$. Compute its Taylor polynomial of degree 2 centered at $x = 1$.

$$g(1) = 1 \ln 1 = 0$$

$$g'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1 \Rightarrow g'(1) = 1$$

$$g''(x) = \frac{1}{x} \Rightarrow g''(1) = 1$$

$$T_2(x) = g(1) + g'(1)(x-1) + \frac{g''(1)}{2!}(x-1)^2 = (x-1) + \frac{1}{2}(x-1)^2$$

- Finding $g(1) = 0, g'(1) = 1, g''(1) = 1$: +2 pts.
- Stating the formula $g(1) + g'(1)(x-1) + \frac{g''(1)}{2!}(x-1)^2$: + 2 pts.
- Getting the Taylor polynomial $(x-1) + \frac{1}{2}(x-1)^2$: +1 pt.

(b) (5 pts) The following table is a list of Taylor series of several functions.

| function | Taylor series |
|---|---|
| $e^x$ | $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ |
| $\ln(1+x)$ | $\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ |
| $\sin x$ | $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$ |
| $\cos x$ | $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ |
| $\tan^{-1} x$ | $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ |

By using it, compute the Taylor polynomial of $f(x) = e^{x^2} \ln(1+2x)$ of degree 3 centered at $x = 0$.

$$\begin{aligned} e^{x^2} \ln(1+2x) &= \left(1 + (x^2) + \frac{(x^2)^2}{2!} + \cdots\right)\left((2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \cdots\right) \\ &= \left(1 + x^2 + \frac{x^4}{2!} + \cdots\right)\left(2x - 2x^2 + \frac{8}{3}x^3 - 4x^4 + \cdots\right) \\ &= 2x - 2x^2 + \frac{8}{3}x^3 + 2x^3 + \cdots \end{aligned}$$

Therefore

$$T_3(x) = 2x - 2x^2 + \frac{14}{3}x^3.$$

- By using Taylor series, getting

$$e^{x^2}\ln(1+2x) = \left(1 + (x^2) + \frac{(x^2)^2}{2!} + \cdots\right)\left((2x) - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \cdots\right)$$

  : 2 pts.

- Computing lowest degree terms and getting $T_3(x) = 2x - 2x^2 + \frac{14}{3}x^3$: 5 pts.

(c) (3 pts) By using (b), find an approximation of $e^{0.01}\ln(1.2)$.

$$e^{0.01}\ln(1.2) = f(0.1) \approx T_3(0.1) = 2(0.1) - 2(0.1)^2 + \frac{14}{3}(0.01)^3 \approx 0.1847$$

(12) (a) (4 pts) *By applying l'Hospital's rule*, evaluate the limit

$$\lim_{x\to 0}\frac{x\sin x}{e^x-1-x}.$$

$$\lim_{x\to 0}\frac{x\sin x}{e^x-1-x}\overset{\frac{0}{0}}{=}\lim_{x\to 0}\frac{\sin x+x\cos x}{e^x-1}\overset{\frac{0}{0}}{=}\lim_{x\to 0}\frac{\cos x+\cos x-x\sin x}{e^x}=\frac{2}{1}=2$$

- Applying l'Hospital's rule and getting $\lim_{x\to 0}\frac{\sin x+x\cos x}{e^x-1}$: 2 pts.
- Applying the rule again and obtaining $\lim_{x\to 0}\frac{\cos x+\cos x-x\sin x}{e^x}$: 3 pts.
- Getting the limit 1: 4 pts.

(b) (4 pts) Evaluate the limit

$$\lim_{x\to 0}\frac{\sin x-\tan^{-1}x-\frac{x^3}{6}}{x\left(\cos x-1+\frac{x^2}{2}\right)}.$$

$$\lim_{x\to 0}\frac{\sin x-\tan^{-1}x-\frac{x^3}{6}}{x\left(\cos x-1+\frac{x^2}{2}\right)}=\lim_{x\to 0}\frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots\right)-\frac{x^3}{6}}{x\left(\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)-1+\frac{x^2}{2}\right)}$$

$$=\lim_{x\to 0}\frac{\frac{x^5}{5!}-\frac{x^5}{5}+\cdots}{\frac{x^5}{4!}+\cdots}=\lim_{x\to 0}\frac{\frac{1}{5!}-\frac{1}{5}+\cdots}{\frac{1}{4!}+\cdots}=\frac{\frac{1}{5!}-\frac{1}{5}}{\frac{1}{4!}}=-\frac{23}{5}$$

- By using power series, getting
$$\lim_{x\to 0}\frac{\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)-\left(x-\frac{x^3}{3}+\frac{x^5}{5}-\cdots\right)-\frac{x^3}{6}}{x\left(\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)-1+\frac{x^2}{2}\right)}$$
: 2 pts.
- Getting the answer $-\frac{23}{5}$: 4 pts.