Homework 1 Solution

submission: Feb 3, in class.

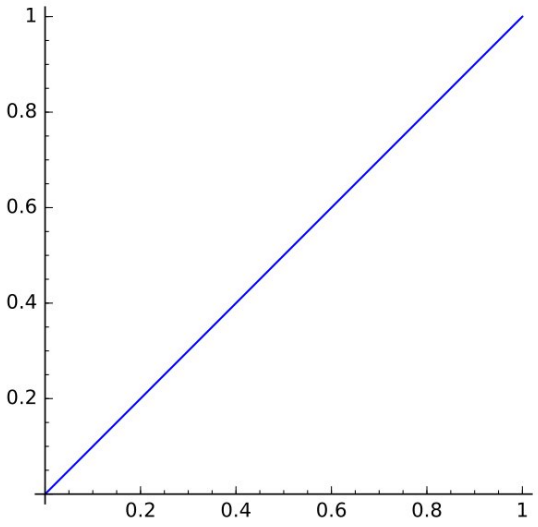
- Write your answer neatly. You have to explain how you deduced your answer. Explain your notations, show computational steps. For each problem, 50 % of the score is for correctness, and 50 % is for neat writing including justification.
- You may discuss with your classmates. But do not copy directly.

1. The *Gini index* is a measure of statistical dispersion intended to represent the income distribution of a nation's residents, and is the most commonly used measure of inequality.[1] Corrado Gini, an Italian economist first devised it in 1912.

   The Gini index is usually defined mathematically based on the *Lorenz curve* $y = L(x)$. L is a function defined on $[0, 1]$ where $L(x)$ is the proportion of the total income of the population that is cumulatively earned by the bottom x of the population. For instance, if the poorest 20% of the population received 13% of the nation's total income, then $L(0.2) = 0.13$. Likewise, if the bottom 90% of the population earned 60% of the total income, then $L(0.9) = 0.6$.

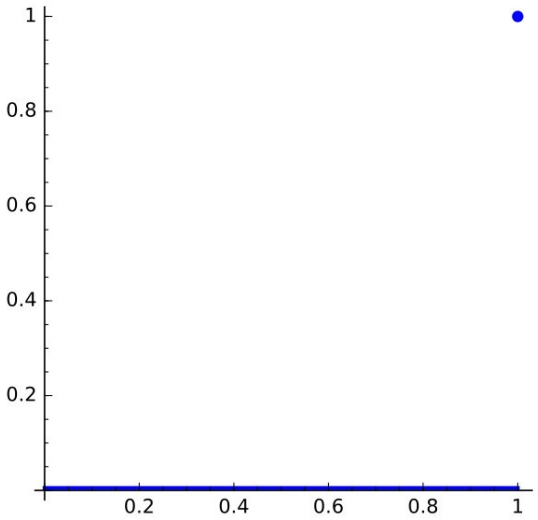
   (a) (8 pts) Plot the Lorentz curve in two extremal cases: 1) A country that everybody has the same income; 2) a country that a single person receives all the income. Explain your answer.

   If everybody has the same income, then the poorest (indeed arbitrary) $x\%$ receive $x\%$ of total income. Therefore $L(x) = x$ and the graph is.



[1] https://en.wikipedia.org/wiki/Gini_coefficient

On the other hand, if a single person receives all the income, then for any $x < 1$, the lowest $100x\%$ of the population has no income. So $L(x) = 0$. If $x = 1$, then $L(1)$ is by definition the ratio of the total income to the total income, $L(1) = 1$.



(b) (10 pts) The Lorenz curve is a curve under the diagonal line $y = x$. The *Gini index* is defined by the area between the Lorenz curve and the line $y = x$ divided by the area under $y = x$. In other words, the Gini index is $A/(A+B)$ in the figure in the next page.

Show that the Gini index G is equal to

$$2\int_0^1 x - L(x)dx.$$

By the area formula,

$$A = \int_0^1 x - L(x)dx.$$

On the other hand, $A + B$ is the area of the isosceles right triangle, which is $1/2$. Therefore

$$\frac{A}{A+B} = \frac{\int_0^1 x - L(x)dx}{1/2} = 2\int_0^1 x - L(x)dx.$$

(c) (6 pts) Evaluate Gini index in two extremal cases in (a).

In the case 1), $L(x) = x$. Thus

$$2\int_0^1 x - L(x)dx = 2\int_0^1 x - xdx = 2\int_0^1 0dx = 0.$$

In the case 2), except the last point $x = 1$, $L(x) = 0$. Thus

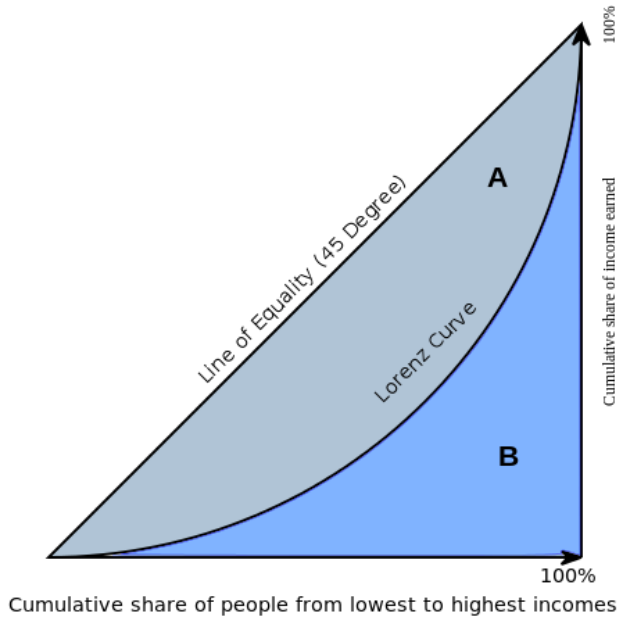
$$2\int_0^1 x - L(x)dx = 2\int_0^1 x - 0dx = 2 \cdot \frac{1}{2}x^2\Big]_0^1 = 1.$$

(d) (10 pts) In a country, the Lorenz curve is given by

$$L(x) = 3x - 2 + 2(1-x)^{3/2}.$$

Find the Gini index.

$$\begin{aligned} G &= 2\int_0^1 x - L(x)dx = 2\int_0^1 x - (3x - 2 + 2(1-x)^{3/2})dx \\ &= 2\int_0^1 -2x + 2 - 2(1-x)^{3/2}dx \\ &= 2\left(-x^2 + 2x + \frac{4}{5}(1-x)^{5/2}\right)\Big]_0^1 \\ &= 2(-1+2) - 2\left(\frac{4}{5}\right) = 0.4 \end{aligned}$$



(e) (6 pts) The below table is the US Gini index over last 40 years.[2]

[2] https://www.census.gov/hhes/www/income/data/historical/inequality/Table%20IE-1.pdf

Year	Gini index
1970	0.394
1975	0.397
1980	0.403
1985	0.419
1990	0.428
1995	0.450
2000	0.462
2005	0.469
2010	0.470

What can we say about the nationwide income inequality? Explain your answer.

(f) (this is not homework) The Lorenz curve is always increasing and concave upward. Explain the reason.

A larger Gini index means the income inequality is larger. From the table, we can see that the nationwide income inequality has become larger over last 40 years.