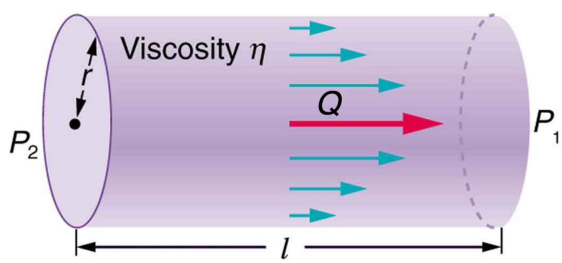
Homework 2 Solution

Due date: February 22nd.

- Write your full name on your homework.
- Write your answer neatly. You have to explain how you deduced your answer. Explain your notations, show computational steps. For each problem, 50 % of the score is for correctness, and 50 % is for neat writing including justification.
- You may discuss with your classmates. But do not copy directly.

1. When we consider the flow of blood through a blood vessel, such as a vein or artery, we can model the shape of the blood vessel by a cylindrical tube with radius r and length ℓ as the figure below:



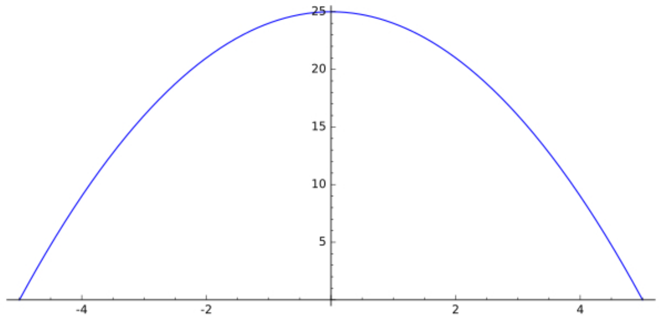
Because of the friction at the walls of the tube, the velocity Q of the blood is greatest along the central axis of the tube and decreases as the distance x from the axis increases until v becomes 0 at the wall. The relationship between x and v is given by the *law of laminar flow* discovered by the French physician Jean-Louis-Marie Poiseuille in 1840. This law states that

$$Q(x) = \frac{P}{4\eta\ell}(r^2 - x^2)$$

where P is the pressure difference ($P_2 - P_1$) between the ends of the tube, η is the viscosity of the blood.

(a) (6 pts) Sketch the graph of $y = Q(x)$ on the interval $[-r, r]$. (Choose your own values of constants P, η, and ℓ.)

Below is an example that $r = 5$, $P = 4$, $\eta = \ell = 1$. In general, it is a upside-down parabola intersecting x-axis at two points $\pm r$.



(b) (10 pts) The *flux* of the blood stream is the volume of blood per unit time that flows across the section of the vessel. By using integral, evaluate the flux and deduce

$$F = \frac{\pi P r^4}{8\eta\ell}.$$

This result is called *Poiseuille's law*.

Note that the graph of $Q(x)$ in (a) describes the blood stream on a slice of a vessel, which contains the central axis of the vessel. Because a blood vessel is a circular cylinder, the flux is the volume of the solid obtained by rotating the region bounded by $y = Q(x)$, x-axis, y-axis, and $y = r$, about y-axis. To find its volume, we can apply the cylindrical shell method. Note that the height of the shell is $Q(x)$.

$$\begin{aligned} F &= \int_0^r 2\pi x Q(x) dx = \int_0^r 2\pi x \frac{P}{4\eta\ell}(r^2 - x^2) dx \\ &= \frac{2\pi P}{4\eta\ell} \int_0^r xr^2 - x^3 dx = \frac{\pi P}{2\eta\ell}\left(\frac{1}{2}x^2 r^2 - \frac{1}{4}x^4\right)\Big]_0^r \\ &= \frac{\pi P}{2\eta\ell}\left(\frac{1}{2}r^4 - \frac{1}{4}r^4\right) = \frac{\pi P}{2\eta\ell}\cdot\frac{1}{4}r^4 = \frac{\pi P r^4}{8\eta\ell} \end{aligned}$$

(c) (8 pts) High blood pressure results from constriction of the arteries. To maintain a normal flux, the heart has to pump harder, thus increasing the blood pressure. In general, the viscosity of the blood η and the length of the vessel ℓ are constants. Use Poiseuille's law derived in (b) to show that if r and P are normal values of the radius and pressure in an artery, and r' and P' are constricted values, then for the flux to remain constant, P' and r' are related by the equation

$$\frac{P'}{P} = \left(\frac{r}{r'}\right)^4.$$

From $F = \dfrac{\pi P r^4}{8\eta\ell}$, we have

$$P = \frac{8F\eta\ell}{\pi r^4}.$$

Then $P' = \dfrac{8F\eta\ell}{\pi r'^4}$. Now

$$\frac{P'}{P} = \frac{\frac{8F\eta\ell}{\pi r'^4}}{\frac{8F\eta\ell}{\pi r^4}} = \frac{8F\eta\ell}{\pi r'^4} \cdot \frac{\pi r^4}{8F\eta\ell} = \frac{r^4}{r'^4} = \left(\frac{r}{r'}\right)^4.$$

(d) (6 pts) By using (c), explain why arteriosclerosis, the thickening and hardening of the walls of arteries is very dangerous.

The small amount of thickening of the walls of arteries results big increment of the blood pressure. For instance, suppose that the radius of arteries is decreased by 25%. Then the new radius (say r') is $\frac{3}{4}r$ where r is the normal radius. Then by using (c), we obtain

$$\frac{P'}{P} = \left(\frac{r}{r'}\right)^4 = \left(\frac{r}{3r/4}\right)^4 = \left(\frac{4}{3}\right)^4 \approx 3.16.$$

This means $P' \approx 3.6P$. So the blood pressure becomes more than three times of the normal blood pressure. Therefore arteriosclerosis is regarded as a major threat for the circulatory system.