Homework 3 Solution

due date: Mar. 3.

- Write your answer neatly. You have to explain how you deduced your answer. Explain your notations, show computational steps. For each problem, 50 % of the score is for correctness, and 50 % is for neat writing including justification.
- You may discuss with your classmates. But do not copy directly.

1. A detective is called to the scene of a crime where a dead body has just been found. She arrives on the scene at 10:23 pm and begins her investigation. Immediately, the temperature of the body is taken and is found to be $80°F$. The detective checks the programmable thermostat and finds that the room has been kept at a constant $68°F$ for the past 3 days.

   (a) (5 pts) Recall that Newton's *Law of Cooling* says that the temperature $T(t)$ of an object in a cool place with a constant surrounding temperature T_s satisfies the differential equation
   $$\frac{dT}{dt} = k(T - T_s)$$
   where k is a constant. Is k positive or negative? Explain your answer.

   The surrounding temperature is lower than that of the object. So the object is cooling, which implies that $\frac{dT}{dt} < 0$. But $T - T_s > 0$, so $k < 0$.

   (b) (7 pts) Derive the formula for $T(t)$ satisfying the equation in (a).

   Because T_s is constant, $\frac{dT_s}{dt} = 0$.
   $$k(T - T_s) = \frac{dT}{dt} = \frac{dT}{dt} = \frac{dT_s}{dt} = \frac{d}{dt}(T - T_s)$$
   Therefore if we set $f := T - T_s$, then f satisfies $\frac{df}{dt} = kf$. Then $f(t) = f(0)e^{kt}$.
   $$T(t) - T_s = (T(0) - T_s)e^{kt}$$
   $$T(t) = T_s + (T(0) - T_s)e^{kt}.$$

   (c) (8 pts) After evidence from the crime scene is collected, the temperature of the body is taken once more and found to be $78.5°F$. This last temperature reading was taken exactly one hour after the first one. Assuming that the victim's body temperature was normal ($98.6°F$) prior to death, find the constant k.

Suppose that t_0 is 10:23 pm. Then $T(t_0) = 80$, and $T(t_0 + 1) = 78.5$. Also $T_s = 68$.

$$78.5 = T(t_0+1) = T_s + (T(0) - T_s)e^{k(t_0+1)} = 68 + (T(0) - 68)e^{kt_0}e^k$$

$$\Rightarrow (T(0) - 68)e^{kt_0}e^k = 10.5$$

$$80 = T(t_0) = T_s(T(0) - T_s)e^{kt_0} = 68 + (T(0) - 68)e^{kt_0}$$

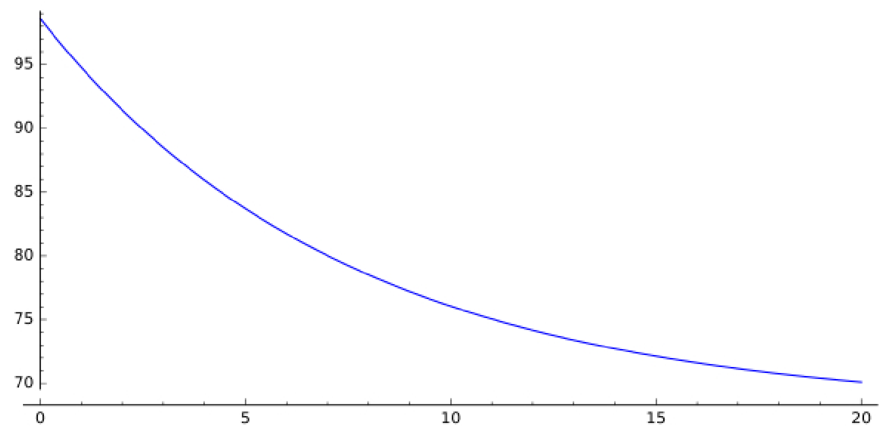
$$\Rightarrow (T(0) - 68)e^{kt_0} = 12$$

$$e^k = \frac{(T(0)-68)e^{kt_0}e^k}{(T(0)-68)e^{kt_0}} = \frac{10.5}{12}$$

$$\Rightarrow k = \ln\frac{10.5}{12}$$

(d) (5 pts) Sketch the graph of $T(t)$.

$$T(t) = 68 + (98.6 - 68)e^{\ln\frac{10.5}{12}t}$$



(e) (8 pts) What time did our victim die? Explain your answer.

$$80 = T(t_0) = 68 + (98.6 - 68)e^{\ln\frac{10.5}{12}t}$$

$$12 = 30.6e^{\ln\frac{10.5}{12}t} \Rightarrow \frac{12}{30.6} = e^{\ln\frac{10.5}{12}t}$$

$$\ln\frac{12}{30.6} = \ln e^{\ln\frac{10.5}{12}t} = \ln\frac{10.5}{12}t$$

$$t = \frac{\ln\frac{12}{30.6}}{\ln\frac{10.5}{12}} \approx 7$$

Therefore approximately the victim died 7 hours prior to the initial investigation, which is 3:23 pm.

(f) (7 pts) When an investigator applies this method to estimate the time of death of a victim for an actual crime scene, what would be realistic obstacles?

The method estimating the time of death depends on several assumptions that are not easy to guarantee. For instance, the surrounding temperature must be a constant, which is not quite common. Also we must know that the vitcim's body temperature while he/she was living.