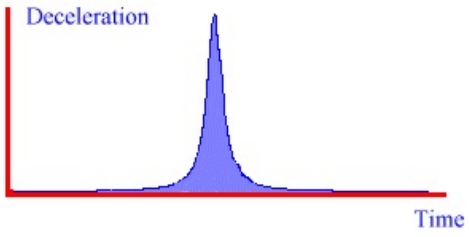
Homework 4 Solution

due date: March 24.

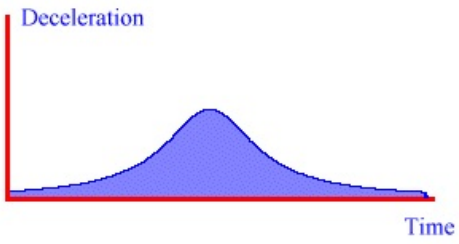
- Write your answer neatly. You have to explain how you deduced your answer. Explain your notations, show computational steps. For each problem, 50 % of the score is for correctness, and 50 % is for neat writing including justification.
- You may discuss with your classmates. But do not copy directly.

1. In this problem, we investigate how one can describe the risk of head injury in a crash by using the average of a function.

   When we stop in a car, the deceleration (negative of ordinary acceleration) can be either abrupt (as in a crash), as follows:



   or more gentle, as in normal braking:



   Either way, the area under the curve is the same, since the velocity we must lose is the same.

   Our head is like a pendulum and so it is the most vulnerable part of our body in a crash. In cars without an airbag, the deceleration is quite violent and lasts a very short time. The *Head Injury Criterion (HIC)* is a measure of the likelihood of head injury arising from an impact. The HIC can be used to assess safety related

to vehicles, personal protective gear, and sport equipment. HIC is defined as the following formula. For any time $i > 0$ after the collision, $H(i)$ is defined by

$$H(i) = 0.015 \left(\frac{1}{0.015} \int_i^{i+0.015} a(t)dt \right)^{2.5},$$

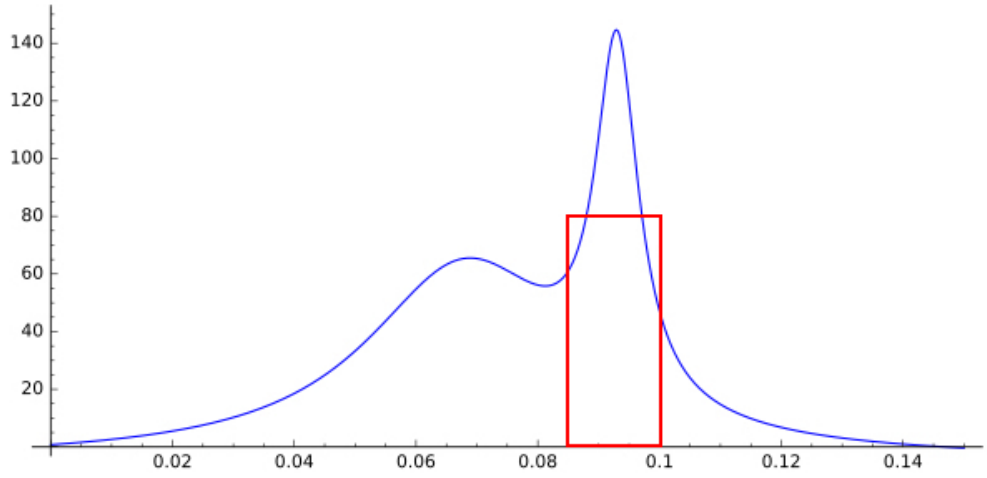
where $a(t)$ is the deceleration (its unit is $g \approx 10\ m/s^2$). Here the unit of the time t is second. In other words, $H(i)$ is the length of the interval $[i, i + 0.015]$ times a power of the *average* deceleration of the interval. The index 2.5 is chosen for the head, based on experiments. Note that by Newton's law, the force affects to the head is proportional to the acceleration (or deceleration). Therefore a larger $H(i)$ gives a more damage to the head.

Now

$$\text{HIC } = \text{Maximum of } H(i),$$

where the maximum is taken over *all* $i > 0$.[1]

(a) (10 pts) Below is a graph of the deceleration of a person's head in a car after a collision when the car does not have an airbag. (The velocity of the car was $45\ mph$.) Which one is larger between $H(0.06)$ and $H(0.1)$? Explain your answer.



On the interval $[0.06, 0.075]$, $a(t) > 50$. So the average deceleration is larger than 50. However, on $[0.1, 0.115]$, $a(t) < 50$. Therefore the average deceleration is less than 50. This implies that

$$H(0.06) > 0.015 \cdot 50^{2.5} > H(0.1).$$

Therefore $H(0.06) > H(0.1)$.

[1] For the actual HIC, the maximum is taken over all intervals $[t_1, t_2]$ whose size is between 0.003 and 0.035.

(b) (10 pts) On the graph above, indicate the value i which makes approximately the largest $H(i)$. (You don't need to find the precise value.) For the i, explain why $H(i) > 800$, so HIC > 800. (You don't need to evaluate any integral. Explain how you obtain the conclusion.)

On the interval $[0.085, 0.1]$, if you look at the red rectangle on the graph with height 80, then it is clear that the area under the graph from $t = 0.085$ to $t = 0.1$ is larger than the area of the rectangle. Therefore

$$\int_{0.085}^{0.1} a(t)dt \geq 80 \cdot 0.015 \Rightarrow \frac{1}{0.015}\int_{0.085}^{0.1} a(t)dt \geq 80.$$

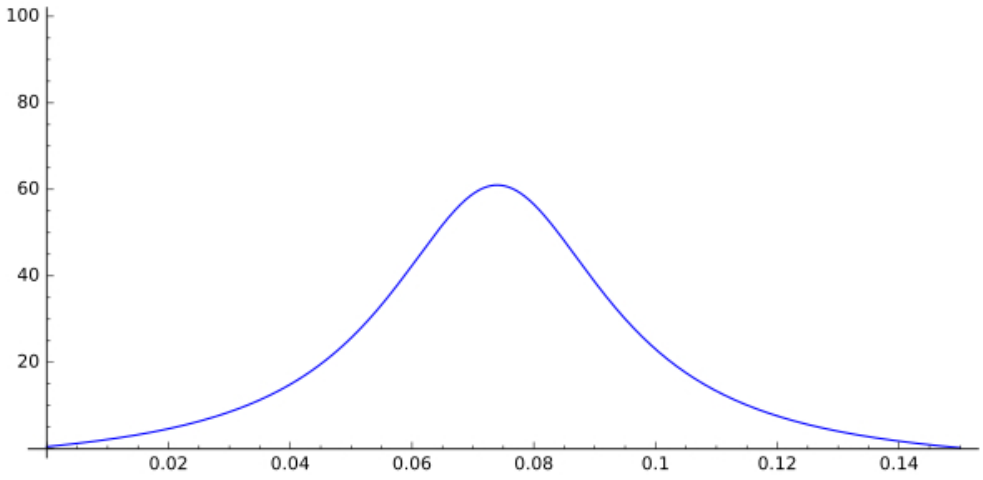
Then

$$H(0.085) = 0.015\left(\frac{1}{0.015}\int_{0.085}^{0.1} a(t)dt\right)^{2.5} \geq 0.015 \cdot 80^{2.5} \approx 858.65.$$

Therefore $H(0.085) > 800$, and HIC $\geq H(0.085) > 800$.

The actual value of HIC that I could evaluate using computer is larger than 1000. Generally, experts agree that HIC values above 1000 are life threatening. At a HIC of 1000, there is an 18% probability of a severe head injury, a 55% probability of a serious injury and a 90% probability of a moderate head injury to the average adult.[2]

(c) (10 pts) Below is a graph of the deceleration of a person's head in a colliding car with the same initial velocity $45\ mph$, but with an airbag. In this case, HIC is less than 450. Explain the reason.



You can see that an airbag effectively reduces HIC and the risk of head injury of drivers.

[2] https://en.wikipedia.org/wiki/Head_injury_criterion

The maximum deceleration is approximately 60 g. Therefore for any interval, the average deceleration on the interval is at most 60. Then

$$H(i) = 0.015 \left(\frac{1}{0.015} \int_i^{i+0.015} a(t)dt \right)^{2.5} \leq 0.015 \cdot 60^{2.5} \approx 418.28 < 450.$$

Therefore HIC, the maximum of $H(i)$, is also less than 450.