Homework 5 Solution

Due date: April 7.

- Write your answer neatly. You have to explain how you deduced your answer. Explain your notations, show computational steps. For each problem, 50 % of the score is for correctness, and 50 % is for neat writing including justification.
- You may discuss with your classmates. But do not copy directly.

1. A (discrete-time) *Markov chain* is a random process that undergoes transitions from one state to another on a state space. It is named after Andrey Markov, a Russian mathematician. It must possess a property that is usually characterized as "memorylessness": the probability distribution of the next state depends only on the current state, not on the states that preceded it. In this homework, we will investigate Markov chains of the simplest type and examples, and their long-term behavior.

   The *Mojave Desert* is a high-desert area that occupies southeastern California. Empirically, the probabilities of weather conditions (modeled as either rainy or sunny), given the weather on the preceding day, is known as the following.

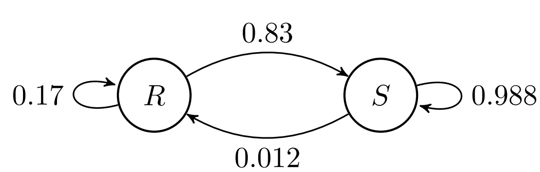
   (a) If one day is rainy, then the probability that the next day is rainy is 17%, and the probability that the next day is sunny is 83%.

   (b) If one day is sunny, then the probability that the next day is rainy is 1.2%, and the probability that the next day is sunny is 98.8%.

   Let R_n be the probability that n-th day is rainy, and let S_n be the probability that n-th day is sunny. Note that $R_n + S_n = 1$ because the sum of probabilities of all possibilities (here we have only two) must be one. We can describe R_{n+1} and S_{n+1} by using above data:

$$\begin{aligned} R_{n+1} &= 0.17R_n + 0.012S_n \\ S_{n+1} &= 0.83R_n + 0.988S_n \end{aligned} \tag{1}$$

   Schematically, we can describe above observation by using a diagram:



(a) (5 pts) Suppose that today, April 7th, 2017, is sunny. So $R_0 = 0$ and $S_0 = 1$. Find the probability that the next day, April 8th, is rainy and the probability that it is sunny. Do the same computation for April 9th.

$$R_1 = 0.17R_0 + 0.012S_0 = 0.17 \cdot 0 + 0.012 \cdot 1 = 0.012$$

$$S_1 = 1 - R_1 = 1 - 0.012 = 0.988$$

Therefore the probability that it is rainy at April 8h is 1.2% and the probability that it is sunny is 98.8%.

$$R_2 = 0.17R_1 + 0.012S_1 = 0.17 \cdot 0.012 + 0.012 \cdot 0.988 \approx 0.0139$$

$$S_2 = 1 - R_2 \approx 1 - 0.0139 = 0.9761$$

Therefore the probability that it is rainy at April 9th is approximately 1.39% and the probability that it is sunny is 97.61%.

(b) (5 pts) By using $R_n + S_n = 1$ and the first equation in (1), find a formula for R_{n+1} in terms of R_n only (without S_n). By using this formula and $R_0 = 0$, evaluate R_n for $1 \le n \le 5$.

$$\begin{aligned} R_{n+1} &= 0.17R_n + 0.012S_n = 0.17R_n + 0.012(1 - R_n) \\ &= (0.17 - 0.012)R_n + 0.012 = 0.158R_n + 0.012 \end{aligned}$$

$$R_1 = 0.158R_0 + 0.012 = 0.158 \cdot 0 + 0.012 = 0.012$$

$$R_2 = 0.158R_1 + 0.012 = 0.158 \cdot 0.012 + 0.012 \approx 0.0139$$

$$R_3 = 0.158R_2 + 0.012 \approx 0.158 \cdot 0.0139 + 0.012 \approx 0.0142$$

$$R_4 = 0.158R_3 + 0.012 \approx 0.158 \cdot 0.0142 + 0.012 \approx 0.0142$$

$$R_5 = 0.158R_4 + 0.012 \approx 0.158 \cdot 0.0142 + 0.012 \approx 0.0142$$

(c) (10 pts) Find a formula of R_n for arbitrary n. Simplify the formula by using the summation formula for a finite geometric series

$$a + ar + ar^2 + \cdots + ar^n = \frac{a(1 - r^{n+1})}{1 - r}.$$

(Hint: Find a pattern.)

$$\begin{aligned} R_2 &= 0.158R_1 + 0.012 = 0.158(0.158R_0 + 0.012) + 0.012 \\ &= 0.158^2 R_0 + 0.158 \cdot 0.012 + 0.012 \end{aligned}$$

$$\begin{aligned} R_3 &= 0.158R_2 + 0.012 = 0.158(0.158^2 R_0 + 0.158 \cdot 0.012 + 0.012) + 0.012 \\ &= 0.158^3 R_0 + 0.158^2 \cdot 0.012 + 0.158 \cdot 0.012 + 0.012 \end{aligned}$$

$$\begin{aligned} R_4 &= 0.158R_3 + 0.012 \\ &= 0.158(0.158^3 R_0 + 0.158 \cdot 0.012^2 + 0.158 \cdot 0.012 + 0.012) + 0.012 \\ &= 0.158^4 R_0 + 0.158^3 \cdot 0.012 + 0.158^2 \cdot 0.012 + 0.158 \cdot 0.012 + 0.012 \end{aligned}$$

So we can see that

$$R_n = 0.158^n R_0 + 0.158^{n-1} \cdot 0.012 + 0.158^{n-2} \cdot 0.012 + \cdots + 0.158 \cdot 0.012 + 0.012$$

The second part of the above sum is a finite geometric series:

$$0.158^{n-1} \cdot 0.012 + 0.158^{n-2} \cdot 0.012 + \cdots + 0.158 \cdot 0.012 + 0.012$$

$$= 0.012 + 0.012 \cdot 0.158 + 0.012 \cdot 0.158^2 + \cdots + 0.012 \cdot 0.158^{n-1}$$
$$= 0.012 \frac{1 - 0.158^n}{1 - 0.158}$$

Therefore we have

$$R_n = 0.158^n R_0 + 0.012 \frac{1 - 0.158^n}{1 - 0.158}.$$

(d) (5 pts) Evaluate

$$\lim_{n \to \infty} R_n.$$

Because $|0.158| < 1$, $\lim_{n \to \infty} 0.158^n = 0$. Therefore

$$\lim_{n \to \infty} R_n = \lim_{n \to \infty} 0.158^n R_0 + 0.012 \frac{1 - 0.158^n}{1 - 0.158} = 0R_0 + 0.012 \frac{1 - 0}{1 - 0.158} \approx 0.0143.$$

So approximately the limit is 0.0143, or 1.43%.

(e) (5 pts) We can interpret this limit in (d) as the probability that a randomly chosen day is rainy. Explain the reason.

For a randomly chosen day, if we regard it as the n-th day for some large n, then the probability the it is rainy is R_n, which is very close to the limit. If we take a larger n, then we can get R_n which is less dependent to the choice of the initial weather R_0, because the term $0.158^n R_0$ approaches 0. Moreover, the limit does not depend on the 'initial condition' R_0. Thus it can be translated as the probability that it is rainy, without regarding any initial condition.