MATH 1207 R04 MIDTERM EXAM 2 SOLUTION

SPRING 2017 - MOON

Name: ______________________________

- Write your answer neatly and show steps.
- Except calculators, any electronic devices including laptops and cell phones are not allowed.

(1) (5 pts) Find the derivative of

$$f(x) = \sin^{-1}(e^x).$$

$$y = \sin^{-1}(e^x)$$

$$u = e^x \Rightarrow y = \sin^{-1} u$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{\sqrt{1-u^2}}e^x = \frac{1}{\sqrt{1-(e^x)^2}}e^x = \frac{e^x}{\sqrt{1-e^{2x}}}$$

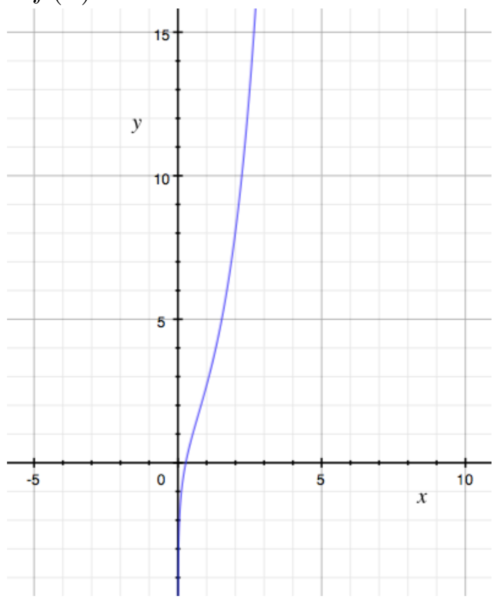
- By using chain rule, getting $\frac{e^x}{\sqrt{1-e^{2x}}}$: 5 pts.

Date: March 10, 2017.

(2) Let $f(x) = e^x + \ln x$.

(a) (2 pts) Find the domain and the range of f. You don't need to write the reason.

Below is the graph of $f(x)$.



domain: $(0, \infty)$
range: $(-\infty, \infty)$

- Finding the domain and range: 1 pt each.

(b) (2 pts) Show that f is one-to-one.

For $x > 0$, $f'(x) = e^x + \dfrac{1}{x} > 0$ because both e^x and $1/x$ are positive. Therefore $f(x)$ is an increasing function and f is one-to-one.

- One must provide a valid reason to get a credit.

(c) (2 pts) Find $f^{-1}(e)$.

$$f^{-1}(e) = x \Leftrightarrow f(x) = e \Leftrightarrow e^x + \ln x = e \Leftrightarrow x = 1$$

Therefore $f^{-1}(e) = 1$.

(d) (2 pts) Find $(f^{-1})'(e)$.

$$(f^{-1})'(e) = \frac{1}{f'(1)} = \frac{1}{e^1 + \frac{1}{1}} = \frac{1}{e+1}$$

- Stating the inverse function theorem correctly: 1 pt.
- Getting the answer $\dfrac{1}{e+1}$: 2 pts.

(3) (9 pts) A fossilized animal bone is unearthed on the ruins of Pompeii. It contains 80% of Carbon-14 found in living matter. About how old is the bone? (It takes 5730 years to be half of the initial amount of Carbon-14.)

Let $m(t)$ be the amount of Carbon-14 in the bone after t years. Then

$$m(t) = Ce^{kt}$$

$$Ce^{5730k} = m(5730) = \frac{1}{2}C \Rightarrow e^{5730k} = \frac{1}{2} \Rightarrow 5730k = \ln\frac{1}{2} \Rightarrow k = \frac{\ln\frac{1}{2}}{5730}$$

$$m(t) = Ce^{\frac{\ln\frac{1}{2}}{5730}t}$$

$$m(t) = 0.8C \Rightarrow 0.8C = Ce^{\frac{\ln\frac{1}{2}}{5730}t} \Rightarrow 0.8 = e^{\frac{\ln\frac{1}{2}}{5730}t}$$

$$\Rightarrow \frac{\ln\frac{1}{2}}{5730}t = \ln 0.8 \Rightarrow t = \frac{5730\ln 0.8}{\ln\frac{1}{2}} \approx 1844.65 \text{ years}$$

- Writing correct general solution $m(t) = Ce^{kt}$ for the amount of Carbon-14: 3 pts.
- Finding $m(t) = Ce^{\frac{\ln\frac{1}{2}}{5730}t}$: 6 pts.
- Writing the answer with an appropriate unit 1844.65 years: 9 pts.
- Writing the answer without stating the unit: -1 pt.

(4) (a) (5 pts) Find the antiderivative

$$\int \frac{6x}{1+x^4} dx.$$

$$u = x^2 \Rightarrow du = 2x dx$$

$$\int \frac{6x}{1+x^4} dx = \int \frac{3}{1+u^2} du = 3\tan^{-1} u + C = 3\tan^{-1}(x^2) + C$$

- Finding an appropriate substitution $u = x^2$: 2 pts.
- Getting the correct answer $3\tan^{-1}(x^2) + C$: 5 pts.

(b) (5 pts) Find the antiderivative

$$\int \frac{6x^3}{1+x^4} dx.$$

$$u = 1 + x^4 \Rightarrow du = 4x^3 dx$$

$$\int \frac{6x^3}{1+x^4} dx = \int \frac{6}{4} \frac{4x^3}{1+x^4} dx = \int \frac{6}{4} \frac{1}{u} du = \frac{6}{4} \ln |u| + C$$
$$= \frac{6}{4} \ln |1+x^4| + C = \frac{3}{2} \ln(1+x^4) + C$$

- Finding an appropriate substitution $u = 1 + x^4$: 2 pts.
- Getting the correct answer $\frac{3}{2} \ln(1+x^4) + C$: 5 pts.

(5) (a) (5 pts) Evaluate the limit

$$\lim_{x\to 0} \frac{e^x - \cos x - \sin x}{7x^2}.$$

$$\lim_{x\to 0} \frac{e^x - \cos x - \sin x}{7x^2} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{e^x + \sin x - \cos x}{14x} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{e^x + \cos x + \sin x}{14} = \frac{2}{14} = \frac{1}{7}$$

- Applying l'Hospital's rule and getting $\lim_{x\to 0} \frac{e^x + \sin x - \cos x}{14x}$: 3 pts.
- Applying l'Hospital's rule again and obtaining the answer $\frac{1}{7}$: 5 pts.

(b) (5 pts) Evaluate the limit

$$\lim_{x\to 0} (1-2x)^{\frac{1}{x}}.$$

$$\ln\left(\lim_{x\to 0}(1-2x)^{\frac{1}{x}}\right) = \lim_{x\to 0} \ln(1-2x)^{\frac{1}{x}} = \lim_{x\to 0} \frac{1}{x}\ln(1-2x) = \lim_{x\to 0} \frac{\ln(1-2x)}{x}$$

$$\overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{\frac{-2}{1-2x}}{1} = -2$$

Thus the answer is e^{-2}.

- Applying natural logarithm: 2 pts.
- By using l'Hospital's rule, finding the limit -2: 4 pts.
- Getting the answer e^{-2}: 5 pts.

(6) (8 pts) After the consumption of an alcoholic beverage, the concentration of alcohol in the bloodstream (blood alcohol concentration, or BAC) surges as the alcohol is absorbed, followed by a gradual decline as the alcohol is metabolized. The function

$$C(t) = 1.35te^{-2.802t}$$

models the BAC of a male, measured in mg/mL, subjects t hours after rapid consumption of $15mL$ of ethanol (corresponding to one alcoholic drink). Find the maximum BAC during the first 3 hours, and indicate when it occurs.

$$C'(t) = 1.35e^{-2.802t} + 1.35te^{-2.802t} \cdot (-2.802) = 1.35e^{-2.802t}(1 - 2.802t)$$

Because $e^{-2.802t} > 0$,

$$C'(t) = 0 \Leftrightarrow 1.35e^{-2.802t}(1 - 2.802t) = 0 \Leftrightarrow 1 - 2.802t = 0 \Leftrightarrow t = \frac{1}{2.802}$$

$$C(\frac{1}{2.802}) = 1.35 \cdot \frac{1}{2.802} e^{-2.802 \cdot \frac{1}{2.802}} = 1.35 \cdot \frac{1}{2.802} e^{-1} \approx 0.1772$$

$$C(0) = 1.35 \cdot 0 \cdot e^0 = 0$$

$$C(3) = 1.35 \cdot 3 \cdot e^{-2.802 \cdot 3} \approx 0.0009$$

Therefore the maximum BAC is 0.1772 mg/mL, and it occurs $\frac{1}{2.802} \approx 0.3569$ hours after the consumption.

- Finding the derivative $C'(t) = 1.35e^{-2.802t}(1 - 2.802t)$: 3 pts.
- Obtaining the critical point $t = \frac{1}{2.802}$: 5 pts.
- Writing the maximum BAC 0.1772 mg/mL with an appropriate unit: 8 pts.
- Omitting the unit: -1 pt.
- Not comparing with values at endpoints: -1 pt.