MATH 1207 R04 MIDTERM EXAM 3 SOLUTION

SPRING 2017 - MOON

Name: ________________________________

- Write your answer neatly and show steps.
- Except calculators, any electronic devices including laptops and cell phones are not allowed.

(1) For this problem, you don't need to explain the reason.

(a) (2 pts) Give an example of a sequence which is bounded but divergent.

$$a_n = (-1)^n$$

(b) (2 pts) Give an example of a sequence which is convergent but not monotonic.

$$b_n = \frac{1}{n}(-1)^n$$

(c) (2 pts) Give an example of a sequence which is not bounded below and not bounded above.

$$c_n = (-2)^n$$

Date: April 12, 2017.

(2) (a) (4 pts) Compute

$$\sum_{n=0}^{\infty} 5\left(\frac{3}{4}\right)^n.$$

$$\sum_{n=0}^{\infty} 5\left(\frac{3}{4}\right)^n = \frac{5}{1-\frac{3}{4}} = \frac{5}{\frac{1}{4}} = 20$$

(b) (5 pts) Evaluate

$$\sum_{n=2}^{\infty} \frac{2\cdot 3^n - 3\cdot 2^n}{5^n}.$$

$$\sum_{n=0}^{\infty} \frac{2\cdot 3^n - 3\cdot 2^n}{5^n} = \sum_{n=0}^{\infty} 2\left(\frac{3}{5}\right)^n - \sum_{n=0}^{\infty} 3\left(\frac{2}{5}\right)^n = \frac{2}{1-\frac{3}{5}} - \frac{3}{1-\frac{2}{5}} = 5-5 = 0$$

$$\sum_{n=2}^{\infty} \frac{2\cdot 3^n - 3\cdot 2^n}{5^n} = \sum_{n=0}^{\infty} \frac{2\cdot 3^n - 3\cdot 2^n}{5^n} - \frac{2\cdot 3^0 - 3\cdot 2^0}{5^0} - \frac{2\cdot 3^1 - 3\cdot 2^1}{5^1}$$
$$= 0 - (-1) - 0 = 1$$

- Describing the given series as a combination $\sum_{n=0}^{\infty} 2\left(\frac{3}{5}\right)^n - \sum_{n=0}^{\infty} 3\left(\frac{2}{5}\right)^n$ of geometric series: 2 pts.
- Getting the sum 0: 3 pts.
- By subtracting the first two terms, getting the answer 1: 5 pts.

(c) (4 pts) Determine the convergence or divergence of the series

$$\sum_{n=1}^{\infty} \frac{3n^2}{7n^2+5}.$$

Explain your answer.

$$\lim_{n\to\infty} \frac{3n^2}{7n^2+5} = \lim_{n\to\infty} \frac{3}{7+\frac{5}{n^2}} = \frac{3}{7} \neq 0$$

Therefore the series is divergent.

- Computing the limit $\lim_{n\to\infty} \frac{3n^2}{7n^2+5} = \frac{3}{7}$: 2 pts.
- Concluding that the series is divergent: 4 pts.

(3) (a) (5 pts) Find the antiderivative

$$\int x^3 \ln x \, dx.$$

$u = \ln x$, $dv = x^3 dx \Rightarrow du = \frac{1}{x}dx$, $v = \frac{x^4}{4}$

$$\int x^3 \ln x \, dx = (\ln x)\frac{x^4}{4} - \int \frac{x^4}{4} \cdot \frac{1}{x} dx$$
$$= \frac{x^4 \ln x}{4} - \int \frac{x^3}{4} \, dx = \frac{x^4 \ln x}{4} - \frac{x^4}{16} + C$$

- Finding appropriate $u = \ln x, dv = x^3 dx$: 2 pts.
- Applying integration by parts formula appropriately and getting $(\ln x)\frac{x^4}{4} - \int \frac{x^4}{4} \cdot \frac{1}{x} dx$: 4 pts.
- Getting the answer $\frac{x^4 \ln x}{4} - \frac{x^4}{16} + C$: 5 pts.

(b) (5 pts) Find the antiderivative

$$\int \tan^{-1} x \, dx.$$

$u = \tan^{-1} x$, $dv = dx \Rightarrow du = \frac{1}{1+x^2}dx$, $v = x$

$$\int \tan^{-1} x \, dx = \tan^{-1} x \cdot x - \int x \cdot \frac{1}{1+x^2} dx = x \tan^{-1} x - \int \frac{x}{1+x^2} dx$$
$$\overset{t=1+x^2, dt=2x dx}{=} x \tan^{-1} x - \int \frac{1}{2t} dt$$
$$= x \tan^{-1} x - \frac{1}{2} \ln |t| + C = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$$

- Finding appropriate $u = \tan^{-1} x, dv = dx$: 1 pt.
- Applying integration by parts formula appropriately and getting $\tan^{-1} x \cdot x - \int x \cdot \frac{1}{1+x^2} dx$: 3 pts.
- By using substitution, getting $x \tan^{-1} x - \int \frac{1}{2t} dt$: 4 pts.
- Getting the answer $x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$: 5 pts.

(4) (6 pts) By using the trapezoidal rule and $n = 5$, find an approximation of the integral

$$\int_2^4 \cos(x^2)\, dx.$$

$n = 5 \Rightarrow \Delta x = \frac{4-2}{5} = 0.4$

$x_0 = 2, x_1 = 2.4, x_2 = 2.8, x_3 = 3.2, x_4 = 3.6, x_5 = 4$

$$\begin{aligned}\int_2^4 \cos(x^2)\, dx &\approx \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)\right)\\ &= \frac{0.4}{2}\left(\cos 2^2 + 2\cos 2.4^2 + 2\cos 2.8^2 + 2\cos 3.2^2 + 2\cos 3.6^2 + \cos 4^2\right)\\ &\approx 0.12495\end{aligned}$$

- Finding appropriate $\Delta x = 0.4$ and x_i: +1 pt each.
- Using the trapezoidal rule and giving the formula

  $$\frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)\right)$$

  : +2 pts.
- Getting the answer 0.12495: +2 pts.

(5) (8 pts) Evaluate the integral

$$\int_e^\infty \frac{1}{x(\ln x)^2} dx.$$

$$\int \frac{1}{x(\ln x)^2} dx \overset{u=lnx, du=\frac{1}{x}dx}{=} \int \frac{1}{u^2} du = \int u^{-2} du = -u^{-1} + C = -\frac{1}{u} + C = -\frac{1}{\ln x} + C$$

$$\int_e^\infty \frac{x}{(\ln x)^2} dx = \lim_{t\to\infty} \int_e^t \frac{x}{(\ln x)^2} dx = \lim_{t\to\infty} -\frac{1}{\ln x}\Big]_e^t = \lim_{t\to\infty} -\frac{1}{\ln x} + \frac{1}{\ln e} = 1$$

- By applying substitution, getting the antiderivative $-\frac{1}{\ln x} + C$: +4 pts.
- Stating the definition $\lim_{t\to\infty} \int_e^t \frac{x(\ln x)^2}{d} x$ of the improper integral: +2 pts.
- Evaluating the limit and getting the answer 1: +2 pts.

(6) (7 pts) Determine whether the following integral is convergent or not.

$$\int_1^\infty \frac{x^2+1}{x^4+x^7}dx.$$

When $x > 1$,

$$\frac{x^2+1}{x^4+x^7} < \frac{x^2+1}{x^7} < \frac{2x^2}{x^7} = \frac{2}{x^5}$$

Thus

$$\int_1^\infty \frac{x^2+1}{x^4+x^7}dx \leq \int_1^\infty \frac{2}{x^5}dx = 2\int_1^\infty \frac{1}{x^5}dx < \infty.$$

Therefore $\int_1^\infty \frac{x^2+1}{x^4+x^7}dx < \infty$.

- Finding a function (for example $\frac{2}{x^5}$) such that it is larger than $\frac{x^2+1}{x^4+x^7}$ and its integral is finite (one has to provide a reasonable justification): 5 pts.
- By applying the comparison test, showing that the given integral is convergent: 7 pts.