MATH 1700 FINAL

SPRING 2015 - MOON

- Write your answer neatly and show steps. If there is no explanation of your answer, then you may not get the credit.
- Except calculators, any electronic devices including laptops and cell phones are not allowed.
- Do not use the graphing function on your calculator.

(1) Consider the following infinite sum
$$3 + \frac{3 \cdot 4}{5} + \frac{3 \cdot 4^2}{5^2} + \frac{3 \cdot 4^3}{5^3} + \frac{3 \cdot 4^4}{5^4} + \cdots .$$

(a) (3 pts) Write the given sum by using sigma notation.
$$\sum_{n=0}^{\infty} 3\left(\frac{4}{5}\right)^n$$

- Using wrong range of index: -1 pt.

(b) (4 pts) Evaluate the sum.
$$\sum_{n=0}^{\infty} 3\left(\frac{4}{5}\right)^n = 3\sum_{n=0}^{\infty}\left(\frac{4}{5}\right)^n = 3 \cdot \frac{1}{1-\frac{4}{5}} = 3 \cdot \frac{1}{\frac{1}{5}} = 3 \cdot 5 = 15$$

- By using the sum of infinite geometric series, obtaining $3 \cdot \frac{1}{1-\frac{4}{5}}$: 3 pts.
- Getting the correct answer 15: 4 pts.

Date: May 4, 2015.

(2) (5 pts) Compute
$$\prod_{n=1}^{1000} \frac{\sqrt{(n+1)^2+1}}{\sqrt{n^2+1}}.$$
$$\prod_{n=1}^{1000} \frac{\sqrt{(n+1)^2+1}}{\sqrt{n^2+1}} = \frac{\sqrt{2^2+1}}{\sqrt{1^2+1}} \cdot \frac{\sqrt{3^2+1}}{\sqrt{2^2+1}} \cdot \frac{\sqrt{4^2+1}}{\sqrt{3^2+1}} \cdots \frac{\sqrt{1001^2+1}}{\sqrt{1000^2+1}}$$
$$= \frac{\sqrt{1001^2+1}}{\sqrt{1^2+1}} = \frac{\sqrt{1002002}}{\sqrt{2}} = \sqrt{501001} \approx 707.8142$$

- Finding the correct answer $\sqrt{501001}$: 5 pts.
- Using a wrong initial or final term: -2 pts each.

(3) (5 pts) Choose *one* of the following terminologies and write its definition as precisely as you can.

- Bifurcation of a fixed point.
- Locally stable fixed point.
- Dense subset of an interval.

For a one-parameter family $x_{n+1} = f_r(x_n)$ of discrete dynamical systems, we say that a fixed point p_r undergoes a bifurcation at $r = r_0$ if it changes from stable to unstable, and either another stable fixed point or a stable 2-cycle emerges from it.

For a discrete dynamical system $x_{n+1} = f(x_n)$, a fixed point p is called *locally stable* if 1) there is an interval I containing p such that for every point x_0 in I, $|x_0 - p| > |x_1 - p| > |x_2 - p| > \cdots$ and 2) $\lim_{n\to\infty} x_n = p$.

For an interval I and a subset J of I, we say that J is a *dense subset* of I if for each x in J and every $\epsilon > 0$, there is y in I such that $|x - y| < \epsilon$.

- If one states an equivalent condition instead of the definition, one can get 3 pts.
- If one misses one of conditions on the definition, -2 pts for each.

(4) Suppose that a contaminated lake is being cleaned by a filtering process. Each week this process is capable of filtering out 16% of all the pollutants present at that time, but another 3 tons of pollutants seep in.

(a) (3 pts) At the beginning, 2000 tons of pollutants were initially in that lake. Find an iterative model describing the amount of pollutants P_n in n-th week.

$$P_{n+1} = P_n - 0.16P_n + 3 = 0.84P_n + 3, \quad P_0 = 2000$$

- Missing the initial condition: -1 pt.

(b) (5 pts) Solve the model in (a) and find the closed formula for P_n.

$$P_n = 0.84^n P_0 + 3\frac{0.84^n - 1}{0.84 - 1}$$

$$\Rightarrow P_n = 2000 \cdot 0.84^n + \frac{3}{0.16}(1 - 0.84^n) = \left(2000 - \frac{3}{0.16}\right)0.84^n + \frac{3}{0.16}$$

- Unless you get the correct answer for (b), you can't get any point for (c) and (d).

(c) (4 pts) When does the remaining amount of the pollutant become less than 50 tons?

$$P_n = 50 \Rightarrow \left(2000 - \frac{3}{0.16}\right)0.84^n + \frac{3}{0.16} = 50$$

$$\Rightarrow \left(2000 - \frac{3}{0.16}\right)0.84^n = 50 - \frac{3}{0.16} \Rightarrow 0.84^n = \frac{50 - \frac{3}{0.16}}{2000 - \frac{3}{0.16}}$$

$$\Rightarrow n \log 0.84 = \log\left(\frac{50 - \frac{3}{0.16}}{2000 - \frac{3}{0.16}}\right) \Rightarrow n = \frac{1}{\log 0.84}\log\left(\frac{50 - \frac{3}{0.16}}{2000 - \frac{3}{0.16}}\right) \approx 23.7991$$

Therefore it takes at least 24 weeks to become less than 50 tons.

- Using a wrong unit: -1 pt.

(d) (3 pts) Describe the asymptotic behavior of P_n.

Because $\lim_{n\to\infty} 0.84^n = 0$,

$$\lim_{n\to\infty}\left(2000 - \frac{3}{0.16}\right)0.84^n + \frac{3}{0.16} = \frac{3}{0.16} = 18.75$$

Therefore the remaining amount of the pollutants approaches 18.75 tons.

(5) Consider the logistic model $P_{n+1} = 2.5P_n\left(1 - \dfrac{P_n}{1000}\right)$.

(a) (3 pts) Find all fixed points.

Let $f(x) = 2.5x\left(1 - \dfrac{x}{1000}\right) = 2.5x - 0.0025x^2$.

$$f(x) = x \Rightarrow 2.5x - 0.0025x^2 = x \Rightarrow 0.0025x^2 - 1.5x = 0$$

$$\Rightarrow x(0.0025x - 1.5) = 0 \Rightarrow x = 0,\ x = \frac{1.5}{0.0025} = 600$$

Therefore there are two fixed points $x = 0, x = 600$.

(b) (4 pts) Determine their stability.

$$f'(x) = 2.5 - 0.005x$$

$$f'(0) = 2.5 \Rightarrow |f'(0)| = 2.5 > 1$$

Therefore 0 is unstable.

$$f'(600) = 2.5 - 0.005 \cdot 600 = -0.5 \Rightarrow |f'(600)| = 0.5 < 1$$

So 600 is stable.

(c) (3 pts) Determine any oscillation around them.

Because $f'(0) = 2.5 > 0$, there is no oscillation around it. On the other hand, $f'(600) = -0.5 < 0$, so there is oscillation around it.

(6) Consider a dynamical system $x_{n+1} = -x_n^3$.

(a) (4 pts) Find all 2-cycles.

Let $f(x) = -x^3$.

$$f(x) = x \Rightarrow -x^3 = x \Rightarrow x^3 + x = x(x^2+1) = 0 \Rightarrow x = 0$$

So there is only one fixed point $x = 0$.

$$f(f(x)) = x \Rightarrow f(-x^3) = x \Rightarrow x^9 = x \Rightarrow x^9 - x = 0$$
$$\Rightarrow x(x^8 - 1) = 0 \Rightarrow x(x^4-1)(x^4+1) = 0$$
$$\Rightarrow x(x^2-1)(x^2+1)(x^4+1) = 0 \Rightarrow x(x-1)(x+1)(x^2+1)(x^4+1) = 0$$
$$\Rightarrow x = 0, 1, -1$$

$x = 0$ is a fixed point. Because $f(1) = -1$ and $f(-1) = 1$, $\{-1, 1\}$ is a 2-cycle.

- Finding the fixed point $x = 0$: +1 pt.
- Finding all solutions $x = 0, 1, -1$ of $f(f(x)) = x$: +2 pts.
- Getting the answer $\{-1, 1\}$: +1 pt.

(b) (4 pts) Determine their stability.

$$f'(x) = -3x^2$$

$$|f'(-1)f'(1)| = |-3 \cdot (-3)| = 9 > 1$$

Therefore the 2-cycle is unstable.

- Evaluating the derivative $f'(x) = -3x^2$: 1 pt.
- Determining the stability: 4 pts.

(7) In this problem, do *not* use your calculator. Unless you show every computational step, you can't get any credit. Let $A = \begin{bmatrix} 3 & -1 \\ 8 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 5 \\ 4 & -2 \end{bmatrix}$.

(a) (2 pts) Compute $3A - 5B$.

$$3A - 5B = 3\begin{bmatrix} 3 & -1 \\ 8 & -3 \end{bmatrix} - 5\begin{bmatrix} 0 & 5 \\ 4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 9 & -3 \\ 24 & -9 \end{bmatrix} - \begin{bmatrix} 0 & 25 \\ 20 & -10 \end{bmatrix} = \begin{bmatrix} 9-0 & -3-25 \\ 24-20 & -9+10 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -28 \\ 4 & 1 \end{bmatrix}$$

(b) (3 pts) Find AB.

$$AB = \begin{bmatrix} 3 & -1 \\ 8 & -3 \end{bmatrix}\begin{bmatrix} 0 & 5 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 3\cdot 0 + (-1)\cdot 4 & 3\cdot 5 + (-1)\cdot(-2) \\ 8\cdot 0 + (-3)\cdot 4 & 8\cdot 5 + (-3)\cdot(-2) \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 17 \\ -12 & 46 \end{bmatrix}$$

(c) (3 pts) Write the definition of the inverse matrix.

For a matrix A, the inverse matrix of A is a matrix A^{-1} such that $AA^{-1} = I = A^{-1}A$.

- If one states only one of two equalities $AA^{-1} = I$ and $A^{-1}A = I$: -1 pt.

(d) (3 pts) Compute A^{-1}.

$$A^{-1} = \frac{1}{3\cdot(-3) - (-1)\cdot 8}\begin{bmatrix} -3 & 1 \\ -8 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 8 & -3 \end{bmatrix}$$

(8) Consider the following homogeneous linear model

$$\begin{aligned} x_{n+1} &= x_n + y_n \\ y_{n+1} &= \frac{1}{2}x_n + \frac{3}{2}y_n, \end{aligned}$$

with $x_0 = 2$, $y_0 = 1$.

(a) (2 pts) Write a matrix equation $\vec{v}_{n+1} = A\vec{v}_n$ of the model.

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}, \quad \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

(b) (4 pts) Find the eigenvalues of A and corresponding eigenvectors of A.

$$A - rI = \begin{bmatrix} 1-r & 1 \\ \frac{1}{2} & \frac{3}{2} - r \end{bmatrix}$$

$$\det(A - rI) = (1-r)(\frac{3}{2} - r) - 1 \cdot \frac{1}{2} = r^2 - \frac{5}{2}r + 1$$

$$r^2 - \frac{5}{2}r + 1 = 0 \Rightarrow (r - \frac{1}{2})(r-2) = 0 \Rightarrow r = \frac{1}{2}, r = 2$$

For $r = \frac{1}{2}$,

$$A - rI = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 1 \end{bmatrix}$$

$(A - rI)\mathbf{v} = \mathbf{0}$ has a nonzero solution $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

For $r = 2$,

$$A - rI = \begin{bmatrix} -1 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$(A - rI)\mathbf{v} = \mathbf{0}$ has a nonzero solution $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

So there are two eigenvalues $r = \frac{1}{2}, 2$ and their corresponding eigenvectors are $\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ respectively.

- Finding the determinant $r^2 - \frac{5}{2}r + 1$ of $A - rI$: 1 pt.
- Getting two eigenvalues $\frac{1}{2}, 2$: 2 pts.
- Obtaining two corresponding eigenvectors $\begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$: 4 pts.
- Finding one eigenvalue and one eigenvector: 2 pts.

(c) (2 pts) Determine the stability of the origin.

Because $|\frac{1}{2}| < 1$ and $|2| > 1$, it is a saddle.

(d) (6 pts) Compute the closed formula for x_n and y_n.

$A = PDP^{-1}$, where

$$P = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}.$$

Then

$$P^{-1} = \frac{1}{2 \cdot 1 - 1 \cdot (-1)} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}.$$

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = A^n \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = PD^nP^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2^n} & 0 \\ 0 & 2^n \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2^n} & 0 \\ 0 & 2^n \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{4}{3} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} \cdot \frac{1}{2^n} \\ \frac{4}{3} \cdot 2^n \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \cdot \frac{1}{2^n} + \frac{4}{3} \cdot 2^n \\ -\frac{1}{3} \cdot \frac{1}{2^n} + \frac{4}{3} \cdot 2^n \end{bmatrix}$$

Therefore

$$x_n = \frac{2}{3} \cdot \frac{1}{2^n} + \frac{4}{3} \cdot 2^n, \quad y_n = -\frac{1}{3} \cdot \frac{1}{2^n} + \frac{4}{3} \cdot 2^n.$$

- Finding P and D so that $A = PDP^{-1}$: + 2 pts.
- Writing the formula $PD^nP^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$: + 2 pts.
- Getting the answer $x_n = \frac{2}{3} \cdot \frac{1}{2^n} + \frac{4}{3} \cdot 2^n, y_n = -\frac{1}{3} \cdot \frac{1}{2^n} + \frac{4}{3} \cdot 2^n$: + 2 pts.

(9) In this problem, do *not* use your calculator. Unless you show every computational step, you can't get any credit. Let $z = -1 + i$, $w = 5 + 3i$.

(a) (2 pts) Compute $z + w$.

$$z + w = (-1 + i) + (5 + 3i) = (-1 + 5) + (1 + 3)i = 4 + 4i$$

(b) (2 pts) Compute zw.

$$zw = (-1 + i)(5 + 3i) = -1 \cdot 5 + (-1) \cdot 3i + 5i + 3i^2 = -5 - 3i + 5i - 3$$
$$= -8 + 2i$$

(c) (3 pts) Evaluate $\frac{z}{w}$.

$$\frac{z}{w} = \frac{-1+i}{5+3i} = \frac{-1+i}{5+3i} \cdot \frac{5-3i}{5-3i} = \frac{-5+3i+5i-3i^2}{25-15i+15i-9i^2}$$
$$= \frac{-2+8i}{34} = -\frac{2}{34} + \frac{8}{34}i$$

- Having the idea to compute the conjugate $5-3i$ to the numerator and denominator: 1 pt.
- Getting the answer $-\frac{2}{34} + \frac{8}{34}i$: 3 pts.

(d) (3 pts) Find the polar coordinate r and θ of z.

$$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$$-1 + i = \sqrt{2}\cos\theta + i\sqrt{2}\sin\theta \Rightarrow \cos\theta = -\frac{1}{\sqrt{2}},\ \sin\theta = \frac{1}{\sqrt{2}}$$

Therefore $\theta = \frac{3\pi}{4}$.

- Getting $r = \sqrt{2}$: 1 pt.
- Finding $\theta = \frac{3\pi}{4}$: 3 pts.

(e) (5 pts) Compute z^{30}.

$$z = r\cos\theta + ir\sin\theta = re^{i\theta}$$

$$z^{30} = \left(re^{i\theta}\right)^{30} = r^{30}e^{i30\theta} = (\sqrt{2})^{30}(\cos 30 \cdot \frac{3\pi}{4} + i\sin 30 \cdot \frac{3\pi}{4})$$
$$= 2^{15}(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) = 2^{15}i$$

- Describing z in terms of a complex exponent $re^{i\theta}$: +1 pt.
- Knowing Euler's identity: + 1 pt.
- By using the law of exponents, obtaining $(\sqrt{2})^{30}(\cos 30 \cdot \frac{3\pi}{4} + i\sin 30 \cdot \frac{3\pi}{4})$: +2 pts.
- Getting the answer $2^{15}i$: +1 pts.

(10) Consider the following prey-predator model:

$$\begin{aligned} P_{n+1} &= P_n - Q_n + 100 \\ Q_{n+1} &= 2P_n + Q_n - 500. \end{aligned}$$

(a) (3 pts) Find the fixed point (P, Q).

$$\begin{aligned} P &= P - Q + 100 \\ Q &= 2P + Q - 500 \end{aligned}$$

$$P = P - Q + 100 \Rightarrow -Q + 100 = 0 \Rightarrow Q = 100$$
$$Q = 2P + Q - 500 \Rightarrow 2P - 500 = 0 \Rightarrow P = 250$$
$$\Rightarrow (P, Q) = (250, 100)$$

(b) (4 pts) Determine the stability of (P, Q). Set $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$.

$$A - rI = \begin{bmatrix} 1-r & -1 \\ 2 & 1-r \end{bmatrix}$$

$$\det(A - rI) = (1-r)^2 - (-1)\cdot 2 = r^2 - 2r + 3$$

$$r^2 - 2r + 3 = 0 \Rightarrow r = \frac{2 \pm \sqrt{(-2)^2 - 4\cdot 3}}{2} = \frac{2 \pm \sqrt{-8}}{2} = 1 \pm \sqrt{2}i$$

$$|1 + \sqrt{2}i| = \sqrt{1^2 + (\sqrt{2})^2} = \sqrt{3} > 1$$

So the fixed point is a source.

- Finding two complex eigenvalues $1 \pm \sqrt{2}$: 2 pts.
- Determining the stability: 4 pts.

(c) (3 pts) Is there any periodic point of this model? Explain your answer.

For a linear dynamical system with complex eigenvalues, a periodic point occurs only if the modulus of a complex eigenvalue is 1. Therefore there is no periodic points.

- If there is no reasonable explanation of the reason, 1 pt.