Homework 10 Solution

Section 4.1 $\sim$ 4.2.

4.1.4. Construct the linear prey-predator model with $k_1 = k_2 = 0$, satisfying:

(a) $r_1 = 1, r_2 = 1, P_0 = 2000, Q_0 = 1000, P_1 = 1800, Q_1 = 2400.$

$$\begin{aligned} P_{n+1} &= P_n - s_1 Q_n \\ Q_{n+1} &= s_2 P_n + Q_n \end{aligned}$$

In particular,

$$\begin{aligned} P_1 &= P_0 - s_1 Q_0 \\ Q_1 &= s_2 P_0 + Q_0 \end{aligned}$$

and

$$\begin{aligned} 1800 &= 2000 - 1000 s_1 \\ 2400 &= 2000 s_2 + 1000. \end{aligned}$$

Then $s_1 = 0.2$, $s_2 = 0.7$. Therefore

$$\begin{aligned} P_{n+1} &= P_n - 0.2 Q_n \\ Q_{n+1} &= 0.7 P_n + Q_n. \end{aligned}$$

(b) $s_1 = 1, s_2 = 1, P_0 = 400, Q_0 = 200, P_1 = 300, Q_1 = 420.$

$$\begin{aligned} P_{n+1} &= r_1 P_n - Q_n \\ Q_{n+1} &= P_n + r_2 Q_n \end{aligned}$$

In particular,

$$\begin{aligned} P_1 &= r_1 P_0 - Q_0 \\ Q_1 &= P_0 + r_2 Q_0 \end{aligned}$$

and

$$\begin{aligned} 300 &= 400 r_1 - 200 \\ 420 &= 400 + 200 r_2. \end{aligned}$$

Then $r_1 = 1.25$, $r_2 = 0.1$. Therefore

$$\begin{aligned} P_{n+1} &= 1.25 P_n - Q_n \\ Q_{n+1} &= P_n + 0.1 Q_n. \end{aligned}$$

4.1.8. Construct the linear competition model that satisfies:

(a) The growth rate of the first species is 1.5, and of the second is 1.25; each is diminished by 0.4 times the population of the other.

From the conditions, $r_1 = 1.5$, $r_2 = 1.25$, $s_1 = s_2 = 0.4$. Therefore

$$\begin{aligned} P_{n+1} &= 1.5P_n - 0.4Q_n \\ Q_{n+1} &= -0.4P_n + 1.25Q_n. \end{aligned}$$

(b) In addition to the characteristics of (a), the first species undergoes immigration at 2500 per step, and the second migrates at 1200 per step.

$$\begin{aligned} P_{n+1} &= 1.5P_n - 0.4Q_n + 2500 \\ Q_{n+1} &= -0.4P_n + 1.25Q_n - 1200. \end{aligned}$$

4.1.20. The system $\begin{cases} x_{n+1} = ax_n - by_n \\ y_{n+1} = bx_n + ay_n \end{cases}$ where $a = \cos\theta$ and $b = \sin\theta$ for some angle θ, has application in computer animation.

(a) Construct the model for $\theta = \pi/6$.

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \ \sin\frac{\pi}{6} = \frac{1}{2}$$

$$\begin{aligned} x_{n+1} &= \frac{\sqrt{3}}{2}x_n - \frac{1}{2}y_n \\ y_{n+1} &= \frac{1}{2}x_n + \frac{\sqrt{3}}{2}y_n \end{aligned}$$

(b) What θ corresponds to the system $\begin{cases} x_{n+1} = (x_n - y_n)/\sqrt{2} \\ y_{n+1} = (x_n + y_n)/\sqrt{2} \end{cases}$?

$$x_{n+1} = \frac{1}{\sqrt{2}}x_n - \frac{1}{\sqrt{2}}y_n$$

Note that

$$\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

Therefore $\theta = \frac{\pi}{4}$.

4.2.8. Find the unique fixed point of

$$\begin{aligned} x_{n+1} &= 0.75x_n + 0.5y_n + 20 \\ y_{n+1} &= 4x_n - y_n + 40. \end{aligned}$$

Suppose that (x, y) is a fixed point. Then

$$\begin{aligned} x &= 0.75x + 0.5y + 20 \\ y &= 4x - y + 40, \end{aligned}$$

or equivalently,

$$\begin{aligned} 0.25x - 0.5y &= 20 \\ -4x + 2y &= 40. \end{aligned}$$

From $x - 2y = 4(0.25x - 0.5y) = 4 \cdot 20 = 80$,

$$-3x = x - 2y + (-4x + 2y) = 80 + 40 = 120 \Rightarrow x = -40$$

$$x - 2y = 80 \Rightarrow -40 - 2y = 80 \Rightarrow y = -60$$

So the fixed point is $(-40, -60)$.

4.2.12. Explain why a unique fixed point does not exist for

$$\begin{aligned} x_{n+1} &= 2x_n + y_n + 2 \\ y_{n+1} &= 3x_n + 4y_n + 9. \end{aligned}$$

A fixed point is a solution (x, y) of the system of linear equations

$$\begin{aligned} x &= 2x + y + 2 \\ y &= 3x + 4y + 9, \end{aligned}$$

or equivalently,

$$\begin{aligned} -x - y &= 2 \\ -3x - 3y &= 9. \end{aligned}$$

Then $0 = -3(-x - y) + (-3x - 3y) = -3 \cdot 2 + 9 = 3$, which is impossible. So there is no solution of this system of linear equations, and it implies that there is no fixed point.

4.2.18. Find the fixed point of

$$\begin{aligned} x_{n+1} &= 2.5x_n \\ y_{n+1} &= 0.8y_n + 2000 \end{aligned}$$

and determine whether it is a sink, source or saddle.

If (x, y) is the fixed point, $x = 2.5x, y = 0.8y + 2000$. So $x = 0, y = 10000$ and the fixed point is $(0, 10000)$.

The general solution of $x_{n+1} = 2.5x_n$ is $x_n = 2.5^n x_0$, and the general solution of $y_{n+1} = 0.8y_n + 2000$ is

$$y_n = 0.8^n y_0 + 2000 \cdot \frac{0.8^n - 1}{0.8 - 1} = 0.8^n y_0 + 10000(1 - 0.8^n).$$

For the initial condition $(1, 10000)$, $(x_n, y_n) = (2.5^n, 0.8^n \cdot 10000 + 10000(1 - 0.8^n)) = (2.5^n, 10000)$ and it diverges from the fixed point as $n \to \infty$. On the other hand, for another initial condition $(0, 0)$, $(x_n, y_n) = (0, 10000(1 - 0.8^n))$ and as $n \to \infty$, it approaches $(0, 10000)$. Therefore the fixed point is a sink.

4.2.22. Determine whether the origin is a sink, source or saddle of

$$\begin{aligned} x_{n+1} &= 3x_n + y_n \\ y_{n+1} &= 2x_n + 3y_n \end{aligned}$$

by iterating and graphing $(x_0, y_0), \cdots, (x_4, y_4)$ in solution space for several choices of initial conditions.

If $(x_0, y_0) = (1, 0)$,

$(x_1, y_1) = (3, 2),\ (x_2, y_2) = (11, 12),\ (x_3, y_3) = (45, 58),\ (x_4, y_4) = (193, 264).$

If $(x_0, y_0) = (0, 1)$,

$(x_1, y_1) = (1, 3),\ (x_2, y_2) = (6, 11),\ (x_3, y_3) = (29, 45),\ (x_4, y_4) = (132, 193).$

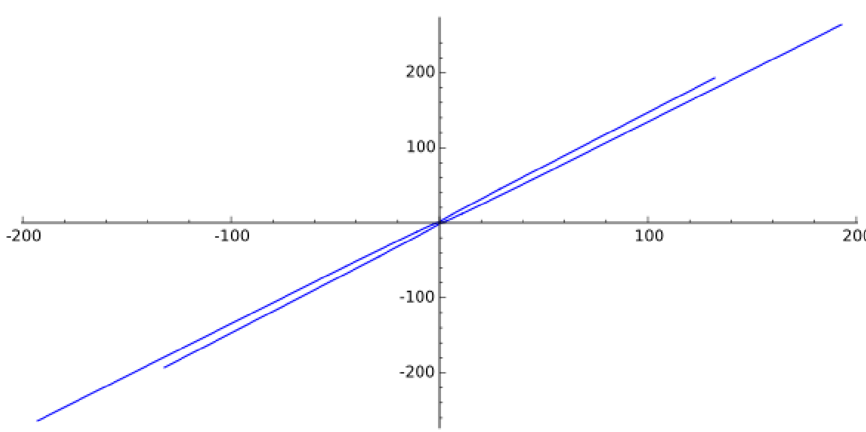
If $(x_0, y_0) = (-1, 0)$,

$(x_1, y_1) = (-3, -2),\ (x_2, y_2) = (-11, -12),\ (x_3, y_3) = (-45, -58),\ (x_4, y_4) = (-193, -264).$

If $(x_0, y_0) = (0, -1)$,

$(x_1, y_1) = (-1, -3),\ (x_2, y_2) = (-6, -11),\ (x_3, y_3) = (-29, -45),\ (x_4, y_4) = (-132, -193).$

The phase graphs of them are the following:



All of the phase graphs are diverging from the origin. Therefore the origin is a source.