Homework 11 Solution

Section 4.3.

4.3.2. For $\mathbf{u} = \begin{bmatrix} -6 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, compute:

(a) $3\mathbf{u}$;

$$3\mathbf{u} = 3\begin{bmatrix} -6 \\ 0 \end{bmatrix} = \begin{bmatrix} 3\cdot(-6) \\ 3\cdot 0 \end{bmatrix} = \begin{bmatrix} -18 \\ 0 \end{bmatrix}$$

(b) $\mathbf{u}+\mathbf{v}$;

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} -6 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -6+5 \\ 0+3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

(c) $-\frac{1}{2}\mathbf{v}$;

$$-\frac{1}{2}\mathbf{v} = -\frac{1}{2}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\cdot 5 \\ -\frac{1}{2}\cdot 3 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} \\ -\frac{3}{2} \end{bmatrix}$$

(d) $2\mathbf{u}-\mathbf{v}$;

$$2\mathbf{u}-\mathbf{v} = 2\begin{bmatrix} -6 \\ 0 \end{bmatrix} - \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2\cdot(-6)-5 \\ 2\cdot 0-3 \end{bmatrix} = \begin{bmatrix} -17 \\ -3 \end{bmatrix}$$

(e) $\frac{1}{3}\mathbf{u}+3\mathbf{v}$.

$$\frac{1}{3}\mathbf{u}+3\mathbf{v} = \frac{1}{3}\begin{bmatrix} -6 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}\cdot(-6)+3\cdot 5 \\ \frac{1}{3}\cdot 0+3\cdot 3 \end{bmatrix} = \begin{bmatrix} 13 \\ 9 \end{bmatrix}$$

4.3.10. For $A = \begin{bmatrix} 2.75 & 1.5 \\ 2.5 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & -3 \\ 1.5 & 2 \end{bmatrix}$, compute:

(a) $5A$;

$$5A = 5\begin{bmatrix} 2.75 & 1.5 \\ 2.5 & -1 \end{bmatrix} = \begin{bmatrix} 5\cdot 2.75 & 5\cdot 1.5 \\ 5\cdot 2.5 & 5\cdot(-1) \end{bmatrix} = \begin{bmatrix} 13.75 & 7.5 \\ 12.5 & -5 \end{bmatrix}$$

(b) $A+B$;

$$\begin{aligned} A+B &= \begin{bmatrix} 2.75 & 1.5 \\ 2.5 & -1 \end{bmatrix} + \begin{bmatrix} -4 & -3 \\ 1.5 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2.75+(-4) & 1.5+(-3) \\ 2.5+1.5 & -1+2 \end{bmatrix} = \begin{bmatrix} -1.25 & -1.5 \\ 4 & 1 \end{bmatrix} \end{aligned}$$

(c) $4A - 2B$.

$$\begin{aligned} 4A - 2B &= 4\begin{bmatrix} 2.75 & 1.5 \\ 2.5 & -1 \end{bmatrix} - 2\begin{bmatrix} -4 & -3 \\ 1.5 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 4\cdot 2.75 - 2\cdot(-4) & 4\cdot 1.5 - 2\cdot(-3) \\ 4\cdot 2.5 - 2\cdot 1.5 & 4\cdot(-1) - 2\cdot 2 \end{bmatrix} = \begin{bmatrix} 19 & 12 \\ 7 & -8 \end{bmatrix} \end{aligned}$$

4.3.12. For $A = \begin{bmatrix} 2 & 0 \\ 5 & 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$, compute:

(a) $A\mathbf{u}$;

$$A\mathbf{u} = \begin{bmatrix} 2 & 0 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 2\cdot 3 + 0\cdot 7 \\ 5\cdot 3 + 1\cdot 7 \end{bmatrix} = \begin{bmatrix} 6 \\ 22 \end{bmatrix}$$

(b) $A\mathbf{v}$.

$$A\mathbf{v} = \begin{bmatrix} 2 & 0 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2\cdot 0 + 0\cdot(-2) \\ 5\cdot 0 + 1\cdot(-2) \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

4.3.16. For $A = \begin{bmatrix} 2.75 & 1.5 \\ 2.5 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & -3 \\ 1.5 & 2 \end{bmatrix}$, compute:

(a) AB;

$$\begin{aligned} AB &= \begin{bmatrix} 2.75 & 1.5 \\ 2.5 & -1 \end{bmatrix}\begin{bmatrix} -4 & -3 \\ 1.5 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2.75\cdot(-4) + 1.5\cdot 1.5 & 2.75\cdot(-3) + 1.5\cdot 2 \\ 2.5\cdot(-4) + (-1)\cdot 1.5 & 2.5\cdot(-3) + (-1)\cdot 2 \end{bmatrix} \\ &= \begin{bmatrix} -8.75 & -5.25 \\ -11.5 & -9.5 \end{bmatrix} \end{aligned}$$

(b) BA;

$$\begin{aligned} BA &= \begin{bmatrix} -4 & -3 \\ 1.5 & 2 \end{bmatrix}\begin{bmatrix} 2.75 & 1.5 \\ 2.5 & -1 \end{bmatrix} \\ &= \begin{bmatrix} -4\cdot 2.75 + (-3)\cdot 2.5 & -4\cdot 1.5 + (-3)\cdot(-1) \\ 1.5\cdot 2.75 + 2\cdot 2.5 & 1.5\cdot 1.5 + 2\cdot(-1) \end{bmatrix} \\ &= \begin{bmatrix} -18.5 & -3 \\ 9.125 & 0.25 \end{bmatrix} \end{aligned}$$

(c) A^2;

$$\begin{aligned} A^2 &= \begin{bmatrix} 2.75 & 1.5 \\ 2.5 & -1 \end{bmatrix} \begin{bmatrix} 2.75 & 1.5 \\ 2.5 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2.75^2 + 1.5 \cdot 2.5 & 2.75 \cdot 1.5 + 1.5 \cdot (-1) \\ 2.5 \cdot 2.75 + (-1) \cdot 2.5 & 2.5 \cdot 1.5 + (-1)^2 \end{bmatrix} \\ &= \begin{bmatrix} 11.3125 & 2.625 \\ 4.375 & 4.75 \end{bmatrix} \end{aligned}$$

(d) A^3.

$$\begin{aligned} A^3 &= A^2 \cdot A = \begin{bmatrix} 11.3125 & 2.625 \\ 4.375 & 4.75 \end{bmatrix} \begin{bmatrix} 2.75 & 1.5 \\ 2.5 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 11.3125 \cdot 2.75 + 2.625 \cdot 2.5 & 11.3125 \cdot 1.5 + 2.625 \cdot (-1) \\ 4.375 \cdot 2.75 + 4.75 \cdot 2.5 & 4.375 \cdot 1.5 + 4.75 \cdot (-1) \end{bmatrix} \\ &= \begin{bmatrix} 37.671875 & 14.34375 \\ 23.90625 & 1.8125 \end{bmatrix} \end{aligned}$$

4.3.26. Find the inverse A^{-1} of $A = \begin{bmatrix} 0 & -1 \\ 2 & -5 \end{bmatrix}$. Verify that $A^{-1}A = I$ and $AA^{-1} = I$.

$$A^{-1} = \frac{1}{0 \cdot (-5) - (-1) \cdot 2} \begin{bmatrix} -5 & -(-1) \\ -2 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -5 & 1 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -\frac{5}{2} & \frac{1}{2} \\ -1 & 0 \end{bmatrix}$$

Indeed,

$$\begin{aligned} A^{-1}A &= \begin{bmatrix} -\frac{5}{2} & \frac{1}{2} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 2 & -5 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{5}{2} \cdot 0 + \frac{1}{2} \cdot 2 & -\frac{5}{2} \cdot (-1) + \frac{1}{2} \cdot (-5) \\ -1 \cdot 0 + 0 \cdot 2 & -1 \cdot (-1) + 0 \cdot (-5) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \end{aligned}$$

and

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} 0 & -1 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} -\frac{5}{2} & \frac{1}{2} \\ -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \cdot (-\frac{5}{2}) + (-1) \cdot (-1) & 0 \cdot \frac{1}{2} + (-1) \cdot 0 \\ 2 \cdot (-\frac{5}{2}) + (-5) \cdot (-1) & 2 \cdot \frac{1}{2} + (-5) \cdot 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \end{aligned}$$

4.3.30. Write the system

$$\begin{aligned} 3x - 6y &= 0 \\ 5x - 9y &= 1 \end{aligned}$$

in vector/matrix form and then use the inverse to solve.

$$\begin{bmatrix} 3 & -6 \\ 5 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -6 \\ 5 & -9 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{3 \cdot (-9) - (-6) \cdot 5} \begin{bmatrix} -9 & 6 \\ -5 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= \frac{1}{3} \begin{bmatrix} 6 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

So $x = 2$, $y = -1$.

4.3.34. (a) Prove that the determinant of a matrix is 0 if and only if one row of the matrix is a multiple of the other.

Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\det A = ad - bc = 0$. First, assume that $a \neq 0$. Then $d = \frac{bc}{a}$. Therefore the second row is

$$\begin{bmatrix} c & d \end{bmatrix} = \begin{bmatrix} \frac{ac}{a} & \frac{bc}{a} \end{bmatrix} = \frac{c}{a} \begin{bmatrix} a & b \end{bmatrix},$$

which is a multiple of the first row. If $a = 0$, $bc = 0$. So $b = 0$ or $c = 0$. If $b = 0$, then the first row is zero row $\begin{bmatrix} 0 & 0 \end{bmatrix}$ and this is obviously a multiple of the second row, namely, $0 \begin{bmatrix} c & d \end{bmatrix}$. If $c = 0$, then the first row is $\begin{bmatrix} 0 & b \end{bmatrix} = b \begin{bmatrix} 0 & 1 \end{bmatrix}$ and the second row is $\begin{bmatrix} 0 & d \end{bmatrix} = d \begin{bmatrix} 0 & 1 \end{bmatrix}$. Therefore one is a multiple of the other.

Conversely, if the first row is a multiple of the second row, then there is r such that $\begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} rc & rd \end{bmatrix}$. Then $\det A = ad - bc = rcd - rdc = 0$. We can check the case that the second row is a multiple of the first row in a similar way.

(b) Prove that the determinant of a matrix is 0 if and only if one column of the matrix is a multiple of the other.

Again, assume that $a \neq 0$. Then $d = \frac{bc}{a}$. Now the second column is

$$\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} \frac{ba}{a} \\ \frac{bc}{d} \end{bmatrix} = \frac{b}{a} \begin{bmatrix} a \\ c \end{bmatrix},$$

which is a multiple of the first column. If $a = 0$, then $bc = 0$ and either $b = 0$ or $c = 0$. If $b = 0$, then the first column is $\begin{bmatrix} 0 \\ c \end{bmatrix} = c \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and the second column is $\begin{bmatrix} 0 \\ d \end{bmatrix} = d \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So one is a multiple of the other. If $c = 0$, then the first column is zero column. Therefore the first column is a multiple (by zero) of the second column.

Conversely, if the first column is a multiple of the second column, then there is r such that $\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} rb \\ rd \end{bmatrix}$. Then $\det A = ad - bc = rbd - brd = 0$. We can check the case that the second row is a multiple of the first row in a similar way.