# Homework 12 Solution

Section 4.4 and 4.8.

4.4.2. Verify that $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ are eigenvectors of $A = \begin{bmatrix} 0.8 & 1.2 \\ 1 & 1 \end{bmatrix}$ and find their corresponding eigenvalues.

$$A\mathbf{u} = \begin{bmatrix} 0.8 & 1.2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -1.2 \\ 1 \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = -\frac{1}{5}\mathbf{u}$$

So $\mathbf{u}$ is an eigenvector with an eigenvalue $-\frac{1}{5}$.

$$A\mathbf{v} = \begin{bmatrix} 0.8 & 1.2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 2\mathbf{v}$$

Therefore $\mathbf{v}$ is an eigenvector with an eigenvalue $2$.

4.4.6. Given the matrix $A = \begin{bmatrix} 1.75 & -1 \\ -0.5 & 1.25 \end{bmatrix}$ and initial vector $\mathbf{x}_0 = \begin{bmatrix} 2500 \\ 2500 \end{bmatrix}$,

(a) Use $\mathbf{x}_{n+1} = A\mathbf{x}_n$ to compute $\mathbf{x}_1$ and $\mathbf{x}_2$;

$$\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} 1.75 & -1 \\ -0.5 & 1.25 \end{bmatrix} \begin{bmatrix} 2500 \\ 2500 \end{bmatrix} = \begin{bmatrix} 1875 \\ 1875 \end{bmatrix} = 0.75 \begin{bmatrix} 2500 \\ 2500 \end{bmatrix}$$

$$\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 1.75 & -1 \\ -0.5 & 1.25 \end{bmatrix} \begin{bmatrix} 1875 \\ 1875 \end{bmatrix} = \begin{bmatrix} 1406.25 \\ 1406.25 \end{bmatrix} = 0.75 \begin{bmatrix} 1875 \\ 1875 \end{bmatrix}$$

(b) Find a formula for $\mathbf{x}_n$.

Because $\mathbf{x}_0$ is an eigenvector of $A$ with eigenvalue $0.75$,

$$\mathbf{x}_n = A^n\mathbf{x}_0 = (0.75)^n\mathbf{x}_0 = (0.75)^n \begin{bmatrix} 2500 \\ 2500 \end{bmatrix}.$$

4.4.8. Compute the eigenvalues of

$$\begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix}.$$

$$A - rI = \begin{bmatrix} 2-r & 2 \\ 1 & 2-r \end{bmatrix}$$

$$\det(A - rI) = (2-r)^2 - 2 \cdot 1 = r^2 - 4r + 2$$

$$\det(A - rI) = 0 \Leftrightarrow r^2 - 4r + 2 = 0 \Leftrightarrow r = \frac{4 \pm \sqrt{8}}{2} = 2 \pm \sqrt{2}$$

4.4.16. Given the matrix $A = \begin{bmatrix} 1/4 & -1/8 \\ -1/2 & 3/4 \end{bmatrix}$, determine whether the origin is a sink, source or saddle of $\mathbf{x}_{n+1} = A\mathbf{x}_n$.

$$A - rI = \begin{bmatrix} \frac{1}{4} - r & -\frac{1}{8} \\ -\frac{1}{2} & \frac{3}{4} - r \end{bmatrix}$$

$$\det(A - rI) = \left(\frac{1}{4} - r\right)\left(\frac{3}{4} - r\right) - \left(-\frac{1}{8}\right)\left(-\frac{1}{2}\right) = r^2 - r + \frac{1}{8}$$

$$r = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot \frac{1}{8}}}{2} = \frac{1 \pm \frac{\sqrt{2}}{2}}{2} = \frac{2 \pm \sqrt{2}}{4}$$

Both of them are positive numbers less than 1. Therefore the origin is a sink.

4.4.22. The eigenvectors of a matrix $A$ are all solutions $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ of $(A - rI)\mathbf{v} = \mathbf{0}$, where $r$ is an eigenvalue. For $A = \begin{bmatrix} 6 & 1 \\ -2 & 3 \end{bmatrix}$, find the eigenvalues and the set of eigenvectors associated with each. Graph each set of eigenvectors.

$$A - rI = \begin{bmatrix} 6 - r & 1 \\ -2 & 3 - r \end{bmatrix}$$

$$\det(A - rI) = (6 - r)(3 - r) - 1 \cdot (-2) = r^2 - 9r + 20$$

$$\det(A - rI) = 0 \Leftrightarrow r^2 - 9r + 20 = 0 \Leftrightarrow (r - 4)(r - 5) = 0 \Leftrightarrow r = 4, 5$$

For the first eigenvalue $r = 4$,

$$A - 4I = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$$

$$(A - 4I)\mathbf{v} = \mathbf{0} \Leftrightarrow \mathbf{v} = \begin{bmatrix} c \\ -2c \end{bmatrix} = c\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

So the eigenvectors are $c\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ for some nonzero real number $c$.

For $r = 5$,

$$A - 5I = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix}$$

$$(A - 5I)\mathbf{v} = \mathbf{0} \Leftrightarrow \mathbf{v} = \begin{bmatrix} c \\ -c \end{bmatrix} = c\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore the corresponding eigenvectors are $c\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for some nonzero real number $c$.

4.4.26. For $A = \begin{bmatrix} 1.75 & -1 \\ -0.5 & 1.25 \end{bmatrix}$,

(a) Find the general form of all solutions $\mathbf{x}_n = \begin{bmatrix} x_n \\ y_n \end{bmatrix}$ of $\mathbf{x}_{n+1} = A\mathbf{x}_n$;

$$A - rI = \begin{bmatrix} 1.75 - r & -1 \\ -0.5 & 1.25 - r \end{bmatrix}$$

$$\det(A - rI) = (1.75 - r)(1.25 - r) - (-1) \cdot (-0.5) = r^2 - 3r + \frac{27}{16}$$

$$r^2 - 3r + \frac{27}{16} = 0 \Leftrightarrow \left(r - \frac{3}{4}\right)\left(r - \frac{9}{4}\right) = 0 \Leftrightarrow r = \frac{3}{4}, \frac{9}{4}$$

$$A - \frac{3}{4}I = \begin{bmatrix} 1 & -1 \\ -0.5 & 0.5 \end{bmatrix}$$

$$(A - \frac{3}{4}I)\mathbf{v} = \mathbf{0} \Rightarrow \mathbf{v} = c\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A - \frac{9}{4}I = \begin{bmatrix} -0.5 & -1 \\ -0.5 & -1 \end{bmatrix}$$

$$(A - \frac{9}{4}I)\mathbf{v} = \mathbf{0} \Rightarrow \mathbf{v} = c\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Set $P = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{9}{4} \end{bmatrix}$. Then

$$P^{-1} = \frac{1}{1 \cdot (-1) - 1 \cdot 2}\begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}.$$

Now

$$\mathbf{x}_n = A^n\mathbf{x}_0 = PD^nP^{-1}\mathbf{x}_0 = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{9}{4} \end{bmatrix}^n\begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \left(\frac{3}{4}\right)^n & 0 \\ 0 & \left(\frac{9}{4}\right)^n \end{bmatrix}\begin{bmatrix} \frac{1}{3}x_0 + \frac{2}{3}y_0 \\ \frac{1}{3}x_0 - \frac{1}{3}y_0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \frac{1}{3}\left(\frac{3}{4}\right)^n x_0 + \frac{2}{3}\left(\frac{3}{4}\right)^n y_0 \\ \frac{1}{3}\left(\frac{9}{4}\right)^n x_0 - \frac{1}{3}\left(\frac{9}{4}\right)^n y_0 \end{bmatrix}$$

$$= \begin{bmatrix} \left(\frac{1}{3}\left(\frac{3}{4}\right)^n + \frac{2}{3}\left(\frac{9}{4}\right)^n\right)x_0 + \left(\frac{2}{3}\left(\frac{3}{4}\right)^n - \frac{2}{3}\left(\frac{9}{4}\right)^n\right)y_0 \\ \left(\frac{1}{3}\left(\frac{3}{4}\right)^n - \frac{1}{3}\left(\frac{9}{4}\right)^n\right)x_0 + \left(\frac{2}{3}\left(\frac{3}{4}\right)^n + \frac{1}{3}\left(\frac{9}{4}\right)^n\right)y_0 \end{bmatrix}$$

(b) Find the unique solution $\mathbf{x}_n$ that satisfies $x_0 = 800$ and $y_0 = 400$;

$$\mathbf{x}_n = \begin{bmatrix} \left(\frac{1}{3}\left(\frac{3}{4}\right)^n + \frac{2}{3}\left(\frac{9}{4}\right)^n\right) 800 + \left(\frac{2}{3}\left(\frac{3}{4}\right)^n - \frac{2}{3}\left(\frac{9}{4}\right)^n\right) 400 \\ \left(\frac{1}{3}\left(\frac{3}{4}\right)^n - \frac{1}{3}\left(\frac{9}{4}\right)^n\right) 800 + \left(\frac{2}{3}\left(\frac{3}{4}\right)^n + \frac{1}{3}\left(\frac{9}{4}\right)^n\right) 400 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1600}{3}\left(\frac{3}{4}\right)^n + \frac{800}{3}\left(\frac{9}{4}\right)^n \\ \frac{1600}{3}\left(\frac{3}{4}\right)^n - \frac{400}{3}\left(\frac{9}{4}\right)^n \end{bmatrix}$$

(c) Compute $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} y_n$ if they exist.

Because $\lim_{n\to\infty}\left(\frac{3}{4}\right)^n = 0$ but $\lim_{n\to\infty}\left(\frac{9}{4}\right)^n = \infty$,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1600}{3}\left(\frac{3}{4}\right)^n + \frac{800}{3}\left(\frac{9}{4}\right)^n = \infty,$$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{1600}{3}\left(\frac{3}{4}\right)^n - \frac{400}{3}\left(\frac{9}{4}\right)^n = -\infty.$$

4.8.2. Find the fixed point of the given system and determine whether it is a sink, source of saddle.

$$\begin{aligned} P_{n+1} &= \frac{2}{3}P_n - \frac{1}{3}Q_n + 4000 \\ Q_{n+1} &= \frac{1}{6}P_n + \frac{1}{6}Q_n + 5000. \end{aligned}$$

If $(P, Q)$ is a fixed point,

$$\begin{aligned} P &= \frac{2}{3}P - \frac{1}{3}Q + 4000 \\ Q &= \frac{1}{6}P + \frac{1}{6}Q + 5000 \end{aligned}$$

$$\begin{aligned} \frac{1}{3}P + \frac{1}{3}Q &= 4000 \\ -\frac{1}{6}P + \frac{5}{6}Q &= 5000 \end{aligned}$$

$$\Rightarrow P = 5000,\ Q = 7000$$

$$A = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix} \Rightarrow A - rI = \begin{bmatrix} \frac{2}{3} - r & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} - r \end{bmatrix}$$

$$\det(A - rI) = \left(\frac{2}{3} - r\right)\left(\frac{1}{6} - r\right) - \left(-\frac{1}{3}\right)\cdot\frac{1}{6} = r^2 - \frac{5}{6}r + \frac{1}{6}$$

$$r^2 - \frac{5}{6}r + \frac{1}{18} = 0 \Leftrightarrow \left(r - \frac{1}{3}\right)\left(r - \frac{1}{2}\right) = 0 \Leftrightarrow r = \frac{1}{3}, \frac{1}{2}$$

Because two eigenvalues are positive numbers less than one, $(P, Q) = (5000, 7000)$ is a sink.

4.8.8. Find the fixed point of the given system and determine whether it is a sink, source of saddle.

$$\begin{aligned} P_{n+1} &= 0.25P_n - Q_n + 2200 \\ Q_{n+1} &= -P_n + 0.25Q_n + 2000. \end{aligned}$$

If $(P, Q)$ is a fixed point,

$$\begin{aligned} P &= 0.25P - Q + 2200 \\ Q &= -P + 0.25Q + 2000 \end{aligned}$$

$$\begin{aligned} 0.75P + Q &= 2200 \\ P + 0.75Q &= 2000 \end{aligned}$$

$$\Rightarrow P = 800,\ Q = 1600$$

$$A = \begin{bmatrix} 0.25 & -1 \\ -1 & 0.25 \end{bmatrix} \Rightarrow A - rI = \begin{bmatrix} 0.25 - r & -1 \\ -1 & 0.25 - r \end{bmatrix}$$

$$\det(A - rI) = (0.25 - r)^2 - (-1)^2 = r^2 - \frac{1}{2}r - \frac{15}{16}$$

$$r^2 - \frac{1}{2}r - \frac{15}{16} = 0 \Leftrightarrow \left(r - \frac{5}{4}\right)\left(r + \frac{3}{4}\right) = 0 \Leftrightarrow r = \frac{5}{4},\ -\frac{3}{4}$$

Because $|\frac{5}{4}| > 1$ and $|-\frac{3}{4}| < 1$, $(P, Q) = (800, 1600)$ is a saddle.