# Homework 1 Solution

Section 2.1

2. Use echelon method to solve each system of two equations in two unknowns. Check your answers.

$$\begin{aligned} 4x + y &= 9 \\ 3x - y &= 5 \end{aligned}$$

Apply the transformation $-\frac{3}{4}R_1 + R_2 \to R_2$:

$$\begin{aligned} 4x + y &= 9 \\ -\frac{7}{4}y &= -\frac{7}{4} \end{aligned}$$

And apply the transformation $\frac{1}{4}R_1 \to R_1$:

$$\begin{aligned} x + \frac{1}{4}y &= \frac{9}{4} \\ -\frac{7}{4}y &= -\frac{7}{4} \end{aligned}$$

Then apply the transformation $-\frac{4}{7}R_2 \to R_2$:

$$\begin{aligned} x + \frac{1}{4}y &= \frac{9}{4} \\ y &= 1 \end{aligned}$$

Then from $x + \frac{1}{4} \cdot 1 = \frac{9}{4}$, we have $x = \frac{9}{4} - \frac{1}{4} = 2$. So the unique solution is $x = 2, y = 1$.

12. Use echelon method to solve each system of two equations in two unknowns. Check your answers.

$$\begin{aligned} 3a - 8b &= 14 \\ a - 2b &= 2 \end{aligned}$$

Apply the transformation $-\frac{1}{3}R_1 + R_2 \to R_2$:

$$\begin{aligned} 3a - 8b &= 14 \\ \frac{2}{3}b &= -\frac{8}{3} \end{aligned}$$

Apply the transformation $\frac{1}{3}R_1 \to R_1$:

$$\begin{aligned} a - \frac{8}{3}b &= \frac{14}{3} \\ \frac{2}{3}b &= -\frac{8}{3} \end{aligned}$$

Finally, apply the transformation $\frac{3}{2}R_2 \to R_2$:

$$\begin{aligned} a - \frac{8}{3}b &= \frac{14}{3} \\ b &= -4 \end{aligned}$$

From $a - \frac{8}{3}(-4) = \frac{14}{3}$, we have $a = -\frac{32}{3} + \frac{14}{3} = -6$. Therefore the unique solution is $a = -6, b = -4$.

16. Use echelon method to solve each system of two equations in two unknowns. Check your answers.

$$\begin{aligned} 3x + 5y + 2 &= 0 \\ 9x + 15y + 6 &= 0 \end{aligned}$$

First of all, write given system of equations as

$$\begin{aligned} 3x + 5y &= -2 \\ 9x + 15y &= -6. \end{aligned}$$

Apply the transformation $-3R_1 + R_2 \to R_2$:

$$\begin{aligned} 3x + 5y &= -2 \\ 0 &= 0 \end{aligned}$$

Then essentially we have only one equation $3x + 5y = -2$, or equivalently, $3x = -5y - 2$. By dividing both sides by 3, we have $x = -\frac{5}{3}y - \frac{2}{3}$. Therefore there are infinitely many solutions and they can be written as $(x, y) = (-\frac{5}{3}y - \frac{2}{3}, y)$.

24. For each of the following systems of equations in echelon form, tell how many solutions there are in nonnegative integers.

$$\begin{aligned} x - 7y + 4z &= 75 \\ 2y + 7z &= 60 \end{aligned}$$

From the second equation, we have $2y = -7z + 60$. By dividing both sides by 2, we have $y = -\frac{7}{2}z + 30$. Plug it in the first equation, we have

$$x - 7(-\frac{7}{2}z + 30) + 4z = 75 \Rightarrow x + \frac{49}{2}z + 4z - 210 = 75 \Rightarrow x = -\frac{57}{2}z + 285.$$

Therefore the solutions can be written as

$$(x, y, z) = (-\frac{57}{2}z + 285, -\frac{7}{2}z + 30, z).$$

Because we want to find nonnegative solutions, we have three inequalities,

$$-\frac{57}{2}z + 285 \ge 0, -\frac{7}{2}z + 30 \ge 0, z \ge 0.$$

$$-\frac{57}{2}z + 285 \ge 0 \Rightarrow 285 \ge \frac{57}{2} \Rightarrow 10 \ge z$$

$$-\frac{7}{2}z + 30 \ge 0 \Rightarrow 30 \ge \frac{7}{2}z \Rightarrow z \le \frac{60}{7} \approx 8.57$$

So the integer values of $z$ satisfying all of those inequalities $z \ge 0$, $z \le 10$, $z \le 8.57$ are $z = 0, 1, 2, 3, 4, 5, 6, 7, 8$.

If $z$ is an odd number, then we can check that $y = -\frac{7}{2}z + 30$ is not an integer. So we need to exclude them. If $z$ is an even number, then it gives integer values of $x = -\frac{57}{2}z + 285$ and $y = -\frac{7}{2}z + 30$. Therefore there are 5 possible $z$ values (0, 2, 4, 6, 8) and so there are 5 solutions.

36. An apparel shop sells skirts for \$45 and blouses for \$35. Its entire stock is worth \$51,750. But sales are slow and only half the skirts and two-thirds of the blouses are sold, for a total of \$30,600. How many skirts and blouses are left in the store?

Suppose that $x$ is the number of blouses in the shop, and $y$ is the number of skirts. If we sell all of them, the revenue is $45x + 35y$. So we have an equation $45x + 35y = 51750$. From the actual selling data, we know that $45 \cdot \frac{1}{2}x + 35 \cdot \frac{2}{3}y = 30600$. Therefore we have a system of linear equations

$$\begin{aligned} 45x + 35y &= 51750 \\ \frac{45}{2}x + \frac{70}{3}y &= 30600. \end{aligned}$$

Apply the transformation $-\frac{1}{2}R_1 + R_2 \to R_2$:

$$\begin{aligned} 45x + 35y &= 51750 \\ \frac{35}{6}y &= 4725 \end{aligned}$$

Apply the transformation $\frac{1}{45}R_1 \to R_1$:

$$\begin{aligned} x + \frac{7}{9}y &= 1150 \\ \frac{35}{6}y &= 4725 \end{aligned}$$

Then apply the transformation $\frac{6}{35}R_2 \to R_2$:

$$\begin{aligned} x + \frac{7}{9}y &= 1150 \\ y &= 810 \end{aligned}$$

Then we have $x + \frac{7}{9} \cdot 810 = 1150$, so $x = 1150 - \frac{7}{9}810 = 520$. Therefore $x = 520$ and $y = 810$.

Note that in this problem, we are interested in remaining numbers of items. Because an half of blouses are already sold, the number of remaining blouses is $520 \cdot \frac{1}{2} = 260$. The number of remaining skirts is $810 \cdot \frac{1}{3} = 270$.

38. Lorri Morgan has $16,000 invested in Disney and Exxon stock. The Disney stock currently sells for $30 a share and the Exxon stock for $70 a share. Her stockbroker points out that if Disney stock goes up 50% and Exxon stock goes up by $35 a share, her stock will be worth $25,500. Is this possible? If so, tell how many shares of each stock she owns. If not, explain why not.

Suppose that $x$ is the number of Disney stocks and $y$ is the number of Exxon stocks she has. Based on the current price, we can make an equation on the total value $30x + 70y = 16000$. If Disney stock goes up 50%, then its new price is $30+0.5\times 30 = 45$. Based on these new prices, the total value becomes $45x+105y = 25500$. Therefore we can make a system of linear equations:

$$\begin{aligned} 30x + 70y &= 16000 \\ 45x + 105y &= 25500 \end{aligned}$$

Apply the transformation $-\frac{45}{30}R_1 + R_2 \to R_2$:

$$\begin{aligned} 30x + 70y &= 16000 \\ 0 &= 1500 \end{aligned}$$

Because the second equation is impossible, there is no solution.

40 A bank teller has a total of 70 bills in five-, ten-, and twenty-dollar denominations. The total value of the money is $960.

(a) Find the total number of solutions.

Suppose that $x$ is the number of five dollar bills, $y$ is that of ten dollar bills, and $z$ is that of twenty dollar bills. From the total number of bills, we can make an equation $x + y + z = 70$. And from the total value, we can find another equation $5x + 10y + 20z = 960$. Therefore we have a system of linear equations

$$\begin{aligned} x + y + z &= 70 \\ 5x + 10y + 20z &= 960. \end{aligned}$$

Apply the transformation $-5R_1 + R_2 \to R_2$:

$$\begin{aligned} x + y + z &= 70 \\ 5y + 15z &= 610 \end{aligned}$$

Apply the transformation $\frac{1}{5}R_2 \to R_2$:

$$\begin{aligned} x + y + z &= 70 \\ y + 3z &= 122 \end{aligned}$$

So we have $y = -3z+122$. From $x+y+z = 70$, we get $x+(-3z+122)+z = 70$, which implies $x = 2z - 52$. Therefore the solutions are given by $(x, y, z) = (2z - 52, -3z + 122, z)$.

So $2z - 52 \geq 0$, $-3z + 122 \geq 0$ and $z \geq 0$.

$$2z - 52 \geq 0 \Rightarrow 2z \geq 52 \Rightarrow z \geq 26$$

$$-3z + 122 \geq 0 \Rightarrow 122 \geq 3z \Rightarrow z \leq \frac{122}{3} \approx 40.66$$

Therefore $z$ is in integer satisfying $z \geq 26$, $z \leq 40.66$ and $z \geq 0$. The possible $z$ values are $z = 26, 27, 28, \cdots, 40$. So there are 15 solutions.

(b) Find the solution with the smallest number of five-dollar bills.

If we minimize the number of five-dollar bills (which is $x$), then we need to take the smallest $z$ value, which is 26. In this case, $(x, y, z) = (2 \cdot 26 - 52, -3 \cdot 26 + 122, 26) = (0, 44, 26)$.

(c) Find the solution with the largest number of five-dollar bills.

To maximize the number of five-dollar bills, we have to take the largest $z$ value, which is 40. Then $(x, y, z) = (2 \cdot 40 - 52, -3 \cdot 40 + 122, 40) = (28, 2, 40)$.