Homework 1 Solution

Section 1.1, 1.2, and 2.1.

1.1.16. Construct a Bouncing Ball Model by determining r and S_0 if the ball is dropped from an initial height of 5.5 feet and its maximum height after each bounce is 10% less then its previous maximum height.

Because S_0 is the initial height, $S_0 = 5.5$. From the second condition, we know that S_{n+1} is 90% of S_n. So $S_{n+1} = 0.9S_n$.

$$S_{n+1} = 0.9S_n, \quad S_0 = 5.5$$

1.1.18. Construct a Bouncing Ball Model by determining r and S_0 if $S_1 = 4.5$ and $S_2 = 2$.

From $S_{n+1} = rS_n$, we know $S_2 = rS_1$.

$$2 = S_2 = rS_1 = 4.5r \Rightarrow r = \frac{2}{4.5} = \frac{4}{9}$$

Also from $S_1 = rS_0$,

$$\frac{4}{9}S_0 = S_1 = 4.5 \Rightarrow S_0 = 4.5 \cdot \frac{9}{4} = \frac{81}{8}.$$

Therefore

$$S_{n+1} = \frac{4}{9}S_n, \quad S_0 = \frac{81}{8}.$$

1.1.20. Use the Bouncing Ball Model to compute $S_1, S_2, \cdots, S_5$ for $r = 0.8$ and $S_0 = 5$.

$$S_1 = rS_0 = 0.8 \cdot 5 = 4$$

$$S_2 = rS_1 = 0.8 \cdot 4 = 3.2$$

$$S_3 = rS_2 = 0.8 \cdot 3.2 = 2.56$$

$$S_4 = rS_3 = 0.8 \cdot 2.56 = 2.048$$

$$S_5 = rS_4 = 0.8 \cdot 2.048 = 1.6384$$

1.2.12. Suppose $1000 is deposited into an account earning simple interest at an annual rate of 3%.

(a) Construct the corresponding Simple Interest Model.

$$r = 0.03, P_0 = 1000 \Rightarrow I = rP_0 = 30$$

$$P_{n+1} = P_n + 30, \quad P_0 = 1000$$

(b) How much will be in the account after 4 years?

$$\begin{aligned} P_4 &= P_3 + 30 = P_2 + 30 + 30 = P_1 + 30 + 30 + 30 = P_0 + 30 + 30 + 30 + 30 \\ &= 1000 + 120 = 1120 \end{aligned}$$

There will be \$1120.

1.2.17. Suppose \$8000 is deposited into an account that earns 6% annual interest compounded monthly.

(a) Construct the corresponding Compound Interest Model.

$$i = \frac{r}{12} = \frac{0.06}{12} = 0.005$$

$$P_{n+1} = (1 + i)P_n = 1.005P_n$$

$$\Rightarrow P_{n+1} = 1.005P_n, \quad P_0 = 8000.$$

(b) Compute the account balance after each of the first 3 months.

$$P_1 = 1.005P_0 = 1.005 \cdot 8000 = \$8040$$

$$P_2 = 1.005P_1 = 1.005 \cdot 8040 = \$8080.2$$

$$P_3 = 1.005P_2 = 1.005 \cdot 8080.2 \approx \$8120.6$$

1.2.18. Suppose a deposit is made into an account that earns interest that is compounded quarterly, and the account balance after 3 months is \$5481.

(a) If the original deposit was \$5416, what is the annual interest rate?

Because the interest is compounded quarterly, P_1 represents the total value after a quarter, which is 3 months. From $P_{n+1} = (1 + i)P_n$,

$$5481 = P_1 = (1 + i)P_0 = (1 + i)5416 \Rightarrow (1 + i) = \frac{5481}{5416} \approx 1.012$$

$$\Rightarrow i \approx 0.012$$

$$r = 4 \cdot i \approx 4 \cdot 0.012 = 0.048$$

Therefore the annual interest rate is approximately 4.8%.

(b) If the annual interest rate is 6%, what was the original deposit?

$$i = \frac{r}{4} = \frac{0.06}{4} = 0.015$$

$$5481 = P_1 = (1+i)P_0 = 1.015P_0$$

$$\Rightarrow P_0 = \frac{5481}{1.015} \approx 5400$$

So the original deposit is approximately \$5400.

1.2.20. Under some circumstances the interest rate i in the Compound Interest Model can be negative. Although this cannot happen for a savings account, it can with riskier investments such as stocks in a declining market.

(a) If $i = -10\%$ and $P_0 = \$1000$, compute P_1, P_2 and P_3.

$$P_1 = (1+i)P_0 = (1-0.1)1000 = \$900$$

$$P_2 = (1+i)P_1 = (1-0.1)900 = \$810$$

$$P_3 = (1+i)P_2 = (1-0.1)810 = \$729$$

(b) For any i satisfying $-1 \le i < 0$, determine the asymptotic behavior of P_n as $n \to \infty$, for any $P_0 \ge 0$.
If $-1 \le i < 0$, then $0 \le 1+i < 1$. Because $P_{n+1} = (1+i)P_n$, the next term P_{n+1} is smaller than P_n. So the total value is decreasing.
Indeed, from

$$P_n = (1+i)P_{n-1} = (1+i)(1+i)P_{n-2} = \cdots (1+i)^n P_0,$$

we know that $P_n = P_0(1+i)^n$. Because $\lim_{n\to\infty}(1+i)^n = 0$ (since $0 \le 1+i < 1$), ultimately P_n approaches 0.

2.1.4. For the price model $P_{n+1} = 100 - P_n$:

(a) If $P_0 = 40$ compute $P_1, \cdots, P_5$.

$$P_1 = 100 - P_0 = 100 - 40 = 60$$

$$P_2 = 100 - P_1 = 100 - 60 = 40$$

$$P_3 = 100 - P_2 = 100 - 40 = 60$$

$$P_4 = 100 - P_3 = 100 - 60 = 40$$

$$P_5 = 100 - P_4 = 100 - 40 = 60$$

(b) If $P_0 = 75$ compute $P_1, \cdots, P_5$.

$$P_1 = 100 - P_0 = 100 - 75 = 25$$

$$P_2 = 100 - P_1 = 100 - 25 = 75$$

$$P_3 = 100 - P_2 = 100 - 75 = 25$$

$$P_4 = 100 - P_3 = 100 - 25 = 75$$

$$P_5 = 100 - P_4 = 100 - 75 = 25$$

(c) Can you predict what will happen for any other value of P_0?

We can prove that regardless the value of P_0, P_n oscillates, in other words, $P_0 = P_2 = P_4 = \cdots$ and $P_1 = P_3 = P_5 = \cdots$.

For $P_{n+1} = 100 - P_n$, by plug $n+1$ in n, we have $P_{n+2} = 100 - P_{n+1}$. The first equation is same to $P_n = 100 - P_{n+1} = P_{n+2}$. Therefore for any n, $P_n = P_{n+2}$.

2.1.6. Construct the Annuity Savings Model that satisfies:

(a) $i = 1\%$, $P_0 = \$1000$, $P_1 = \$1125$.

$$P_1 = (1+i)P_0 + d \Rightarrow 1125 = (1.01)1000 + d = 1010 + d$$

$$\Rightarrow d = 1125 - 1010 = 115$$

So the annuity saving model is

$$P_{n+1} = (1 + 0.01)P_n + 115, \quad P_0 = 1000.$$

(b) $P_0 = \$10,000$, $P_1 = \$12,400$, $P_2 = \$14,836$.

$$P_1 = (1+i)P_0 + d \Rightarrow 12400 = (1+i)10000 + d \Rightarrow 2400 = 10000i + d$$

$$P_2 = (1+i)P_1 + d \Rightarrow 14836 = (1+i)12400 + d \Rightarrow 2436 = 12400i + d$$

So we have a system of linear equations

$$\begin{aligned} 10000i + d &= 2400 \\ 12400i + d &= 2436. \end{aligned}$$

By solving this, we have $i = 0.015$, $d = 2250$. So the annuity saving model is

$$P_{n+1} = (1 + 0.015)P_n + 2250, \quad P_0 = 10000.$$

2.1.12. Construct the Linear Population Model that satisfies:

(a) The initial population is 3500 and the ratio of the next population to the present one is 6/5.
The second condition is simply saying that the growth ratio is 6/5.

$$P_{n+1} = \frac{6}{5}P_n, \quad P_0 = 3500.$$

(b) The initial population is 3500 and the population doubles every 3 generations.
The second condition is saying that $P_3 = 2P_0$.

$$P_3 = rP_2 = rrP_1 = rrrP_0 = P_0r^3$$
$$2P_0 = P_0r^3 \Rightarrow r^3 = 2 \Rightarrow r = 2^{\frac{1}{3}}$$
$$P_{n+1} = 2^{\frac{1}{3}}P_n, \quad P_0 = 3500$$

2.1.14. Construct the Linear Harvesting Model that satisfies:

(a) The population, initially 100,000, is decreasing by 5% per generation and harvesting is occurring at a constant rate of 1500 per generation.
The growth ratio r is $1 - 0.05 = 0.95$. So the population model is

$$P_{n+1} = 0.95P_n - 1500, \quad P_0 = 100000.$$

(b) $P_0 = 75,000$, $P_1 = 77,000$, $P_2 = 79,500$.

$$P_1 = rP_0 + k \Rightarrow 77000 = 75000r + k$$
$$P_2 = rP_1 + k \Rightarrow 79500 = 77000r + k$$

By solving this system of linear equations, we have $r = 1.25$, $k = -16750$. Therefore the corresponding population model is

$$P_{n+1} = 1.25P_n - 16750, \quad P_0 = 75000$$

2.1.18. Suppose that a contaminated body of water is being cleaned by a filtering process. Each week this process is capable of filtering out a certain fraction a of all the pollutants present at that time, but another b tons of pollutants seep in. Here, a and b are constants that satisfy $0 < a < 1$ and $b > 0$.

(a) If T_0 tons of pollutants are initially in that body of water, write a general iterative equation for T_n, the amount of pollutants present n weeks later.

$$T_{n+1} = (1 - a)T_n + b$$

(b) If each week 10% of all pollutants present can be removed but another 2 tons seep in, find the values of a and b and write the iterative equation for this process.
From the assumption, we have $a = 0.1$ and $b = 2$. So we have:

$$T_{n+1} = 0.9T_n + 2$$