Homework 2 Solution

Section 2.2 and 2.3.

2.2.4. Write each of the following autonomous equations in the form $x_{n+1} = ax_n + b$ and identify the constants a and b.

(a) $x_{n+1} = 2(4 - x_n)/3$.

$$x_{n+1} = \frac{2(4-x_n)}{3} = \frac{8-2x_n}{3} = -\frac{2x_n}{3} + \frac{8}{3} = -\frac{2}{3}x_n + \frac{8}{3}$$

$$\Rightarrow x_{n+1} = -\frac{2}{3}x_n + \frac{8}{3}$$

$$a = -\frac{2}{3}, \quad b = \frac{8}{3}$$

(b) $3x_{n+1} - 2x_n = 8$.

$$3x_{n+1} - 2x_n = 8 \Rightarrow 3x_{n+1} = 2x_n + 8 \Rightarrow x_{n+1} = \frac{2}{3}x_n + \frac{8}{3}$$

$$a = \frac{2}{3}, \quad b = \frac{8}{3}$$

(c) $(x_{n+1} + x_n)/2 = 1$.

$$\frac{x_{n+1} + x_n}{2} = 1 \Rightarrow x_{n+1} + x_n = 2 \Rightarrow x_{n+1} = -x_n + 2$$

$$a = -1, \quad b = 2$$

(d) $x_{n+1}/x_n = 0.9$.

$$\frac{x_{n+1}}{x_n} = 0.9 \Rightarrow x_{n+1} = 0.9x_n$$

$$a = 0.9, \quad b = 0$$

2.2.6. Write

$$x_1 = x_0, x_2 = 2x_1 + 1, x_3 = 3x_2 + 2, x_4 = 4x_3 + 3, \cdots.$$

in the form $x_{n+1} = a_n x_n + b_n$, and then compute and make a time-series plot of $x_0, x_1, \cdots, x_5$ using $x_0 = 1$.

If we use the general form $x_{n+1} = a_n x_n + b_n$ of a linear iterative model, then

$$a_0 = 1, a_1 = 2, a_2 = 3, a_3 = 4, \cdots \Rightarrow a_n = n + 1$$

and

$$b_0 = 0, b_1 = 1, b_2 = 2, b_3 = 3, \cdots \Rightarrow b_n = n.$$

So $x_{n+1} = (n+1)x_n + n$.

Now

$$x_1 = x_0 = 1,$$

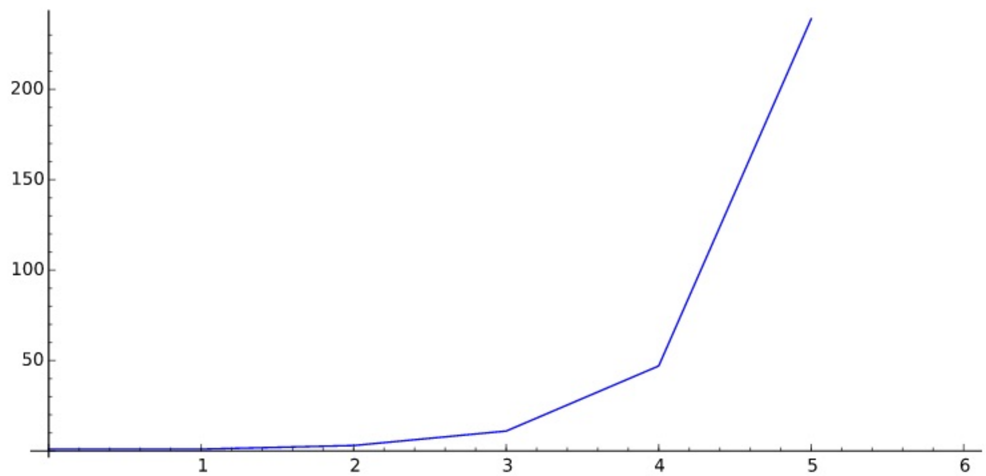
$$x_2 = 2x_1 + 1 = 2 \cdot 1 + 1 = 3,$$

$$x_3 = 3x_2 + 2 = 3 \cdot 3 + 2 = 11,$$

$$x_4 = 4x_3 + 3 = 4 \cdot 11 + 3 = 47,$$

$$x_5 = 5x_4 + 4 = 5 \cdot 47 + 4 = 239.$$

The following graph is the time-series graph of x_n.



2.2.8. Write the following iteration

$$x_1 = 1 - \frac{x_0}{2}, x_2 = 1 + \frac{x_1}{2}, x_3 = 1 - \frac{x_2}{2}, x_4 = 1 + \frac{x_3}{2}, \cdots .$$

in the form $x_{n+1} = a_n x_n + b_n$, and then compute and make a time-series plot of $x_0, x_1, \cdots, x_5$ using $x_0 = 1$.

$$a_0 = -\frac{1}{2}, a_1 = \frac{1}{2}, a_2 = -\frac{1}{2}, a_3 = \frac{1}{2}, \cdots \Rightarrow a_n = \frac{(-1)^{n+1}}{2}$$

$$b_0 = 1, b_1 = 1, b_2 = 1, b_3 = 1, \cdots \Rightarrow b_n = 1$$

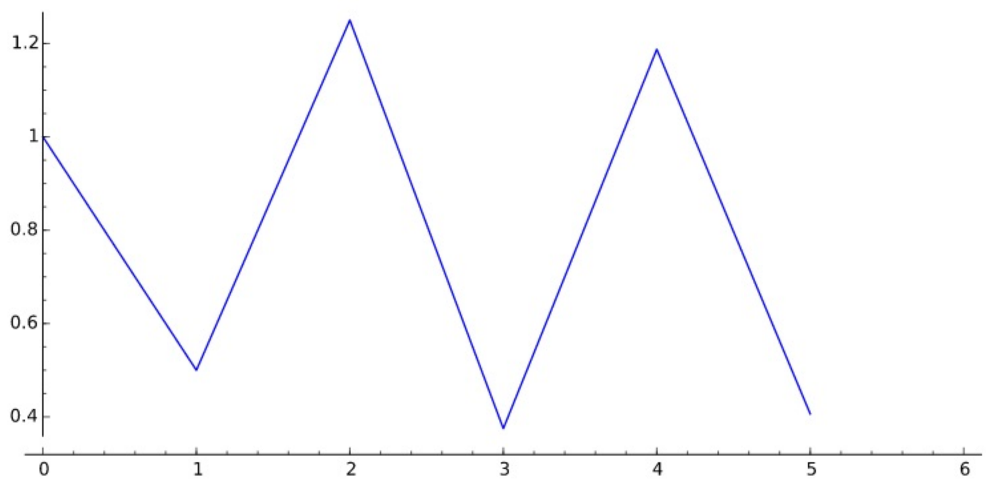
Therefore $x_{n+1} = \frac{(-1)^{n+1}}{2} x_n + 1$.

So

$$x_1 = -\frac{1}{2}x_0 + 1 = -\frac{1}{2} \cdot 1 + 1 = \frac{1}{2},$$

$$x_2 = \frac{1}{2}x_1 + 1 = \frac{1}{2} \cdot \frac{1}{2} + 1 = \frac{5}{4},$$
$$x_3 = -\frac{1}{2}x_2 + 1 = -\frac{1}{2} \cdot \frac{5}{4} + 1 = \frac{3}{8},$$
$$x_4 = \frac{1}{2}x_3 + 1 = \frac{1}{2} \cdot \frac{3}{8} + 1 = \frac{19}{16},$$
$$x_5 = -\frac{1}{2}x_4 + 1 = -\frac{1}{2} \cdot \frac{19}{16} + 1 = \frac{13}{32}.$$

The time-series graph is the below.



2.2.18. Construct an iterative method of computing the partial sums S_n and identify which N would make S_N equal the entire sum for

$$\sum_{k=1}^{100} \frac{-k}{2^k}.$$

Let $a_k = \frac{-k}{2^k}$. We would like to compute $\sum_{k=1}^{100} a_k$.

Note that S_n, the partial sum is the sum of first n terms. Thus $S_0 = 0$ and

$$S_{n+1} = S_n + a_{n+1} = S_n + \frac{-(n+1)}{2^{n+1}} = S_n + \frac{-n-1}{2^{n+1}}.$$

And $\sum_{k=1}^{100} \frac{-k}{2^k} = S_{100}$.

2.2.20. Construct an iterative method of computing the partial sums S_n and identify which N would make S_N equal the entire sum for

$$1 + 4 + 9 + 16 + 25 + \cdots + 10,000.$$

Note that this is a sum of squares, that is,

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \cdots + 100^2.$$

If we define $a_k = k^2$, then the sum we want to find is

$$\sum_{k=1}^{100} a_k = \sum_{k=1}^{100} k^2.$$

Let $S_0 = 0$ and

$$S_{n+1} = S_n + a_{n+1} = S_n + (n+1)^2.$$

The 100th term $S_{100} = \sum_{k=1}^{100} k^2$ is what we want to find.

2.2.24. Suppose someone borrows $10,000 at an annual interest rate of 6% compounded monthly and makes monthly payments on the loan. Construct an iterative model for the monthly unpaid balance if the monthly payments are:

(a) $1, $2, $4, $8, $16, $\cdots$.

The monthly interest rate i is $0.06/12 = 0.005$. Then from the Loan Payment Model, if we denote the remaining balance after n-th month as P_n,

$$P_1 = (1+i)P_0 - 1 = 1.005P_0 - 1,$$

$$P_2 = (1+i)P_1 - 2 = 1.005P_1 - 2,$$

$$P_3 = (1+i)P_2 - 4 = 1.005P_2 - 4,$$

$$P_4 = (1+i)P_3 - 8 = 1.005P_3 - 8,$$

etc. If we denote the general iterative equation as $P_{n+1} = 1.005P_n - d_n$, then $d_0 = 1, d_1 = 2, d_2 = 4 = 2^2, d_3 = 8 = 2^3, \cdots$. Therefore $d_n = 2^n$ and

$$P_{n+1} = 1.005P_n - 2^n, \quad P_0 = 10000.$$

(b) $100, $200, $100, $200, $100, $\cdots$.

By using the same idea, if we denote the general iteration as $P_{n+1} = 1.005P_n - d_n$, then $d_0 = 100, d_1 = 200, d_3 = 100, d_4 = 200, \cdots$. We are able to describe d_n as

$$d_n = 150 + (-1)^{n+1}50.$$

Then $P_{n+1} = 1.005P_n - (150 + (-1)^{n+1}50) = 1.005P_n - 150 - (-1)^{n+1}50$. Thus

$$P_{n+1} = 1.005P_n - 150 - (-1)^{n+1}50, \quad P_0 = 10000$$

is the model describing the remaining balance.

2.3.6. If $P_0 = 1$, construct an iterative equation for computing the partial products P_n, identify which N would make P_N equal the entire product, and compute $P_1, \cdots, P_5$ for

$$\left(1-\frac{1}{2}\right)\cdot\left(1+\frac{2}{4}\right)\cdot\left(1-\frac{3}{8}\right)\cdot\left(1+\frac{4}{16}\right)\cdot\cdots\cdot\left(1+\frac{20}{2^{20}}\right).$$

If we define a sequence

$$a_1 = 1-\frac{1}{2}, a_2 = 1+\frac{2}{4}, a_3 = 1-\frac{3}{8}, \cdots,$$

then a formula for the n-th term is

$$a_n = 1+(-1)^n\frac{n}{2^n}.$$

Then

$$P_n = \prod_{k=1}^{n} a_n = \prod_{k=1}^{n}\left(1+(-1)^n\frac{n}{2^n}\right),$$

the product of the first n-term has the iteration

$$P_{n+1} = a_{n+1}P_n = \left(1+(-1)^{n+1}\frac{n+1}{2^{n+1}}\right)P_n,$$

with $P_0 = 1$. Also P_{20} is the product we want to compute.

Finally,

$$P_1 = \left(1-\frac{1}{2}\right)P_0 = \frac{1}{2},$$

$$P_2 = \left(1+\frac{2}{4}\right)P_1 = \frac{6}{4}\cdot\frac{1}{2} = \frac{3}{4},$$

$$P_3 = \left(1-\frac{3}{8}\right)P_2 = \frac{5}{8}\cdot\frac{3}{4} = \frac{15}{32},$$

$$P_4 = \left(1+\frac{4}{16}\right)P_3 = \frac{5}{4}\cdot\frac{15}{32} = \frac{75}{128},$$

$$P_5 = \left(1-\frac{5}{32}\right)P_4 = \frac{27}{32}\cdot\frac{75}{128} = \frac{2025}{4096}.$$

2.3.8. For

$$x_{n+1} = \sqrt{\frac{n+2}{n+1}}x_n,$$

the solution x_n with $x_0 = 1$ is a product that telescopes to a relatively simple expression. Find that expression.

For a homogeneous equation $x_{n+1} = a_n x_n$, the closed formula is

$$x_n = x_0 \prod_{k=0}^{n-1} a_k.$$

So

$$x_n = \prod_{k=1}^{n} \sqrt{\frac{k+1}{k}} = \sqrt{\frac{2}{1}} \cdot \sqrt{\frac{3}{2}} \cdot \sqrt{\frac{4}{3}} \cdot \cdots \cdot \sqrt{\frac{n+1}{n}}$$
$$= \sqrt{\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \cdots \cdot \frac{n+1}{n}} = \sqrt{\frac{n+1}{1}} = \sqrt{n+1}.$$

2.3.12. Find a formula for the exact solution x_n for

$$x_{n+1} = 2x_n/3, \quad x_0 = 5.$$

Because $x_{n+1} = \frac{2}{3} x_n$ with $x_0 = 5$,

$$x_n = x_0 \left(\frac{2}{3}\right)^n = 5 \cdot \left(\frac{2}{3}\right)^n.$$

2.3.18. Suppose that an investment of \$1000 grows to \$1500 in 2 years getting interest that is compounded monthly.

(a) What is the interest rate?

Suppose that r is the annual interest rate. Then $i = r/12$, because it is compounded monthly. The total value follows the compound interest model $P_{n+1} = (1+i)P_n$, which is an autonomous homogeneous equation.

From the condition given in the problem, after 24 months (= 2 years), $P_0 = 1000$ becomes 1500. So

$$1500 = P_{24} = P_0(1+i)^{24} = 1000(1+i)^{24}.$$

Thus

$$(1+i)^{24} = \frac{1500}{1000} = 1.5$$
$$1 + i = (1.5)^{\frac{1}{24}} \Rightarrow i = (1.5)^{\frac{1}{24}} - 1 \approx 0.017$$
$$r = 12i = 12 \cdot 0.017 \approx 0.204$$

Therefore the annual interest rate is approximately 20.4%.

(b) How much interest (in dollars) does the investment earn during the first year?

The total interest in the first year can be computed by $P_{12} - P_0$.

$$P_{12} - P_0 = P_0(1+i)^{12} - P_0 = 1000 \cdot (1.017)^{12} - 1000 \approx 224.20$$

So the total interest is approximately \$224.20.

2.3.22. Suppose that when a certain radioactive substance decays, its mass M_n on day n satisfies $M_{n+1} = aM_n$. Assuming its mass is measured to be 250 grams initially and 100 grams on day 7:

(a) Find the formula for the mass M_n.

The iterative equation is autonomous homogeneous. So we know that $M_n = M_0 a^n$. Also from the assumption, $M_0 = 250$ and $M_7 = 100$. So

$$100 = M_7 = M_0 a^7 = 250a^7 \Rightarrow a^7 = \frac{100}{250} = \frac{2}{5} \Rightarrow a = \left(\frac{2}{5}\right)^{\frac{1}{7}}.$$

Therefore

$$M_n = 250 \cdot a^n = 250 \left(\left(\frac{2}{5}\right)^{\frac{1}{7}}\right)^n = 250 \left(\frac{2}{5}\right)^{\frac{n}{7}}.$$

(b) What is the approximate half-life of the substance, i.e., the time its takes to lose half its mass?

We want to find n such that $M_n = M_0/2$.

$$M_n = \frac{M_0}{2} \Rightarrow 250\left(\frac{2}{5}\right)^{\frac{n}{7}} = \frac{250}{2} \Rightarrow \left(\frac{2}{5}\right)^{\frac{n}{7}} = \frac{1}{2} \Rightarrow \frac{n}{7}\log\left(\frac{2}{5}\right) = \log\left(\frac{1}{2}\right)$$

$$\Rightarrow n = 7\frac{\log\left(\frac{1}{2}\right)}{\log\left(\frac{2}{5}\right)} \approx 5.295$$

So approximately it takes 5.3 days (the answer '5 days' is also ok).

2.3.24. Suppose that when a certain pendulum is set in motion, it swings in such a way that the greatest positive (or negative) angle it makes on one side of the vertical is always 95% of the greatest negative (or positive) angle it previously made on the other side of the vertical, which always occurs 3 seconds earlier. If the initial angle it makes with the vertical when let go is $20°$:

(a) How many times will the pendulum cross the vertical before the magnitude of the angle it achieves is $1°$ or less?

Suppose that A_n is the angle from the vertical axis. Then from the assumption, $A_{n+1} = -0.95A_n$. Note that in each step, the sign does change because the pendulum moves to the opposite side. Then $A_n = A_0(-0.95)^n = 20(-0.95)^n$.

We want to find the smallest n such that $|A_n| \leq 1$, because we are interested in the magnitude of the angle, not its direction.

$$|A_n| = |20(-0.95)^n| = 20(0.95)^n$$

$$|A_n| \leq 1 \Rightarrow 20(0.95)^n \leq 1 \Rightarrow 0.95^n \leq \frac{1}{20}$$

$$\Rightarrow n \log 0.95 \leq \log\left(\frac{1}{20}\right) \Rightarrow n \geq \frac{\log\left(\frac{1}{20}\right)}{\log 0.95} \approx 58.4$$

(Note that at the last step, the direction of the inequality does change because $\log 0.95$ is a negative number.)

Because n is an integer, the smallest n with the property is 59.

(b) Approximately how long will it take for this to happen?

Although the swing distance decreases, a swing always takes 3 seconds. So the answer is $59 \times 3 = 177$ seconds.