Homework 3 Solution

Section 2.4.

2.4.4. For the geometric series

$$1-0.2+0.04-0.008+0.0016-\cdots-(0.2)^{15},$$

identify a and n, and determine the sum S.

$$\begin{aligned}&1-0.2+0.04-0.008+0.0016-\cdots-(0.2)^{15}\\ &=1+(-0.2)+(-0.2)^2+(-0.2)^3+(-0.2)^4+\cdots+(-0.2)^{15}\end{aligned}$$

So $a=-0.2$ and $n=15$.

$$S=\frac{a^{n+1}-1}{a-1}=\frac{(-0.2)^{16}-1}{(-0.2)-1}=\frac{1-(-0.2)^{16}}{1+0.2}=\frac{1-(0.2)^{16}}{1.2}$$

2.4.8. Evaluate

$$-3+\frac{3}{1.1}-\frac{3}{1.21}+\frac{3}{1.331}-\frac{3}{1.4641}+\cdots$$

or say if it diverges.

$$\begin{aligned}&-3+\frac{3}{1.1}-\frac{3}{1.21}+\frac{3}{1.331}-\frac{3}{1.4641}+\cdots\\ &=-3\left(1-\frac{1}{1.1}+\frac{1}{1.21}-\frac{1}{1.331}+\frac{1}{1.4641}-\cdots\right)\\ &=-3\left(1+\left(-\frac{1}{1.1}\right)+\left(-\frac{1}{1.1}\right)^2+\left(-\frac{1}{1.1}\right)^3+\left(-\frac{1}{1.1}\right)^4+\cdots\right)\\ &=-3\left(\frac{1}{1-(-\frac{1}{1.1})}\right)=-3\left(\frac{1}{\frac{1.2}{1.1}}\right)=-3\left(\frac{1.1}{1.2}\right)=-\frac{3.3}{1.2}\end{aligned}$$

2.4.12. Find a formula for the exact solution x_n for the given x_0.

$$x_{n+1}=\frac{x_n}{4}+5,\quad x_0=3$$

Because it is an autonomous equation with $a=\dfrac{1}{4}$ and $b=5$,

$$\begin{aligned}x_n&=a^nx_0+b\frac{a^n-1}{a-1}=\left(\frac{1}{4}\right)^n3+5\frac{\left(\frac{1}{4}\right)^n-1}{\left(\frac{1}{4}\right)-1}\\ &=\frac{3}{4^n}+5\frac{1-\left(\frac{1}{4}\right)^n}{1-\left(\frac{1}{4}\right)}=\frac{3}{4^n}+5\frac{\frac{4^n-1}{4^n}}{\frac{3}{4}}=\frac{3}{4^n}+\frac{20(4^n-1)}{3\cdot 4^n}\end{aligned}$$

2.4.16. Suppose that each month \$200 is deposited into a savings account with an original balance of \$1500, that pays 4% annual interest compounded monthly.

(a) What will the account balance be 2 years later?

By Annuity Savings Model, the total balance after n months can be described as $P_{n+1} = (1+i)P_n + d$. From the assumption, $i = \dfrac{0.04}{12} = \dfrac{0.01}{3} = \dfrac{1}{300}$, $P_0 = 1500$, and $d = 200$. We want to find P_{24}.

$$\begin{aligned} P_{24} &= (1+i)^{24}P_0 + d\frac{(1+i)^{24}-1}{i} \\ &= \left(1+\frac{1}{300}\right)^{24} 1500 + 200\frac{\left(1+\frac{1}{300}\right)^{24}-1}{\frac{1}{300}} \\ &= 1500\left(\frac{301}{300}\right)^{24} + 60000\left(\left(\frac{301}{300}\right)^{24} - 1\right) \approx 6613.29 \end{aligned}$$

Therefore the balance is \$6613.29.

(b) How much interest will the account earn during that time?

The total interest is the difference between the total balance of the account and the money we have put.

$$P_{24} - (P_0 + 24 \cdot 200) = 6613.29 - (1500 + 4800) = 313.29$$

So the total interest is \$313.29.

2.4.18. Suppose someone borrows \$10,000 at an annual interest rate of 9% to be paid back monthly.

(a) If \$150 is paid each month, what will the unpaid balance be after 3 years?

By Loan Payment Model, the remaining balance follows

$$P_{n+1} = (1+i)P_n - d.$$

In our situation, $P_0 = 10000$, $d = 150$, and $i = \dfrac{0.09}{12} = \dfrac{3}{400}$. We want to find P_{36}.

$$\begin{aligned} P_{36} &= \left(1+\frac{3}{400}\right)^{36} 10000 - 150\frac{\left(1+\frac{3}{400}\right)^{36}-1}{\frac{3}{400}} \\ &= 10000\left(\frac{403}{400}\right)^{36} - 20000\left(\left(\frac{403}{400}\right)^{36} - 1\right) \approx 6913.55 \end{aligned}$$

So the remaining balance is \$6913.55.

(b) How much interest is paid during this time?

The actual amount we've paid back is $150 \cdot 36 = 5400$. This has been used to pay some interest and the balance. The amount we've used to reduce the balance is $P_0 - P_{36} = 10000 - 6913.55 = 3086.45$. Therefore the interest we paid is $5400 - 3086.45 = \$2313.55$.

2.4.20. Suppose that a population is presently 150,000 and is growing by 2% per generation. Immigration has also been occurring at a constant rate of 2500 per generation.

(a) What will the population equal five generations from now?

The population follows the following immigration model $P_{n+1} = rP_n + k$, with $r = 1.02$, $k = 2500$, and $P_0 = 150000$.

$$P_5 = (1.02)^5 150000 + 2500 \frac{(1.02)^5 - 1}{1.02 - 1} \approx 178622$$

(b) What must have been the population fiver generation ago?

We may interpret this question as the following two ways: 1) The immigration model $P_{n+1} = rP_n + k$ can be applied to negative n and find P_{-5}. 2) Set $P_5 = 150000$ and find P_0.

I will follow the second idea. Then

$$150000 = P_5 = (1.02)^5 P_0 + 25000 \frac{(1.02)^5 - 1}{1.02 - 1}$$

$$P_0 = \frac{150000 - 25000 \frac{(1.02)^5 - 1}{1.02 - 1}}{(1.02)^5} \approx 124076$$