Homework 4 Solution

Section 2.5.

2.5.2. If an account earns 9% annual interest compounded monthly, which will yield more at the end of the year: (i) A single deposit of \$1000 at the beginning of the year, or (ii) Deposits of \$100 at the end of every month of the year?

The monthly interest rate is $\frac{0.09}{12} = 0.0075$. In case (i), the total value at at the end of the first year (after 12th term) will be

$$(1 + 0.0075)^{12} \cdot 1000 \approx 1093.81.$$

On the other hand, in case (2), the total value satisfies a discrete dynamical system $P_{n+1} = (1 + 0.0075)P_n + 100$ and $P_0 = 0$. Then

$$P_{12} = 100 \cdot \frac{(1 + 0.0075)^{12} - 1}{0.0075} \approx 1250.76.$$

Therefore the second option yields more.

2.5.4. If \$5000 is borrowed for 2 years at an annual interest rate of 12% paid monthly:

(a) What are the monthly payments?

The monthly interest rate is $\frac{0.12}{12} = 0.01$. From Loan Payment Model $P_{n+1} = (1 + 0.01)P_n - d$, we have

$$P_n = (1 + 0.01)^n P_0 - d\frac{(1 + 0.01)^n - 1}{0.01} = 5000(1.01)^n - d\frac{(1.01)^n - 1}{0.01}.$$

Because we have to pay it back in 2 years (24 months), $P_{24} = 0$.

$$0 = P_{24} = 5000(1.01)^{24} - d\frac{(1.01)^{24} - 1}{0.01}$$

$$\Rightarrow d\frac{(1.01)^{24} - 1}{0.01} = 5000(1.01)^{24}$$

$$\Rightarrow d = 5000(1.01)^{24} \cdot \frac{0.01}{(1.01)^{24} - 1} \approx 235.37$$

Therefore approximately the monthly payment should be \$235.37.

(b) How much interest is paid during this time?

The actual total payment is

$$24d = 24 \cdot 5000(1.01)^{24} \cdot \frac{0.01}{(1.01)^{24} - 1} \approx 5648.82$$

Therefore the total paid interest is $5648.82 - 5000 = \$648.82$. (This value may be slightly different because of rounding.)

2.5.6. Suppose a couple with a combined annual income of \$72,000 would like to purchase their first home. They have \$50,000 available as a down payment and can get a mortgage for the rest at 8% annual interest paid monthly for 30 years. However, the lender will not allow their monthly mortgage payment to exceed 1/4 of their monthly income.

(a) What is the maximum price home they can afford under these conditions?

The monthly income is $72000/12 = 6000$. So their maximum monthly mortgage payment is $6000/4 = 1500$. From Loan Payment Model,

$$P_n = (1+i)^n P_0 - d\frac{(1+i)^n - 1}{i}.$$

Note that $d = 1500$ and $i = \dfrac{0.08}{12}$. Because $P_{30\cdot 12} = P_{360} = 0$,

$$\left(1 + \frac{0.08}{12}\right)^{360} P_0 - 1500\frac{\left(1+\frac{0.08}{12}\right)^{360} - 1}{\frac{0.08}{12}} = 0$$

$$\Rightarrow \left(1 + \frac{0.08}{12}\right)^{360} P_0 = 1500 \cdot 12\frac{\left(1+\frac{0.08}{12}\right)^{360} - 1}{0.08}$$

$$\Rightarrow P_0 = 1500 \cdot 12\frac{\left(1+\frac{0.08}{12}\right)^{360} - 1}{0.08 \cdot \left(1+\frac{0.08}{12}\right)^{360}} \approx 204425$$

Therefore the maximum price is $204425 + 50000 = \$254425$.

(b) What would they have to get their annual income up to in order to afford a \$300,000 home?

In this case, the couple should borrow \$250000 as mortgage. Then the monthly payment d satisfies

$$\left(1 + \frac{0.08}{12}\right)^{360} \cdot 250000 - d\frac{\left(1+\frac{0.08}{12}\right)^{360} - 1}{\frac{0.08}{12}} = 0.$$

$$\left(1 + \frac{0.08}{12}\right)^{360} \cdot 250000 = d\frac{\left(1+\frac{0.08}{12}\right)^{360} - 1}{\frac{0.08}{12}}$$

$$\Rightarrow d = \left(1 + \frac{0.08}{12}\right)^{360} \cdot 250000 \cdot \frac{\frac{0.08}{12}}{\left(1+\frac{0.08}{12}\right)^{360} - 1} \approx 1834.41$$

Therefore their annual income should be $1834.41 \cdot 4 \cdot 12 \approx \$88052$.