Homework 5 Solution

Section 2.6 $\sim$ 2.7.

2.6.4. Find the fixed point and determine both its stability and whether or not oscillation occurs for

$$x_{n+1} = (x_n + 1)/2.$$

From

$$x_{n+1} = \frac{1}{2}x_n + \frac{1}{2},$$

$a = \frac{1}{2}$ and $b = \frac{1}{2}$. The fixed point is

$$\frac{\frac{1}{2}}{1 - \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1.$$

Because $|a| < 1$, the fixed point is stable. Since $a > 0$, there is no oscillation around the fixed point.

2.6.8. Find the fixed point and determine both its stability and whether or not oscillation occurs for

$$x_{n+1} = (7 - 3x_n)/2.$$

In this case

$$x_{n+1} = -\frac{3}{2}x_n + \frac{7}{2},$$

so $a = -\frac{3}{2}$, $b = \frac{7}{2}$. The fixed point is

$$\frac{\frac{7}{2}}{1 - \left(-\frac{3}{2}\right)} = \frac{\frac{7}{2}}{\frac{5}{2}} = \frac{7}{5}.$$

Because $|a| = \frac{3}{2} > 1$, the fixed point is unstable. Also because $a < 0$, the solution is oscillating around the fixed point.

2.6.12. Find the equilibria of the model $P_{n+1} = 0.95P_n + 7.25$. Determine both stability and whether or not oscillation occurs.

There is one equilibrium

$$\frac{7.25}{1 - 0.95} = 145.$$

Because $|0.95| < 1$, the fixed point is stable, and it is positive, thus there is no oscillation.

2.6.14. Find the equilibria of the model in Exercise 23 of Section 2.4. Determine both stability and whether or not oscillation occurs.

In Exercise 23 of Section 2.4, the amount of PCB's in n-th month is described by

$$x_{n+1} = x_n - 0.1x_n + 0.5 = 0.9x_n + 0.5, \quad x_0 = 10.$$

The fixed point is

$$\frac{0.5}{1-0.9} = 5.$$

Because $|0.5| < 1$, the fixed point is stable. Also because it is positive, there is no oscillation.

2.6.16. Suppose that each day during flu season, 15% of those who have the flu in a certain town recover from it, while another 600 people come down with the flu. If there are currently 1800 cases of the flu:

(a) How many cases will there be two weeks from now?

The number of infected people I_n in n-th day can be described as

$$I_{n+1} = I_n - 0.15I_n + 600 = 0.85I_n + 600, \quad I_0 = 1800.$$

So

$$I_n = 0.85^n I_0 + 600\frac{0.85^n - 1}{0.85 - 1}.$$

$$I_{14} = 0.85^{14} \cdot 1800 + 600\frac{0.85^{14} - 1}{0.85 - 1} \approx 3774.$$

Therefore after two weeks, approximated 3774 people are infected.

(b) At what level will the number of cases eventually stabilize?

The fixed point is $\frac{600}{1-0.85} = 4000$. So eventually, the number of cases will be stabilized at 4000.

2.6.18. Suppose that each year 2% of all the trees in a certain forest are destroyed naturally. Also each year 5000 mature trees are harvested for lumber, but 7500 new trees are either planted or sprout up on their own.

(a) Write an iterative equation for the yearly number of trees T_n in the forest.

$$T_{n+1} = T_n - 0.02T_n - 5000 + 7500 = 0.98T_n + 2500$$

(b) If there are currently estimated to be 100,000 trees in that forest, what is the maximum number of trees the forest will ever have?

Because 0.98 has absolute value less than one, the fixed point

$$\frac{2500}{1-0.98} = 125000$$

is stable. Also because $0.98 > 0$, a solution x_n is monotonic. Finally, the initial condition $x_0 = 100000$ is less than 125000, the solution is increasing. Therefore the maximum number of trees is 125,000.

2.7.2. Find the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ for

$$f(x, y) = (2x + 3y)^4.$$

$$f_x(x, y) = 4(2x + 3y)^3 \cdot 2 = 8(2x + 3y)^3$$

$$f_y(x, y) = 4(2x + 3y)^3 \cdot 3 = 12(2x + 3y)^3$$

2.7.4. Find the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ for

$$f(x, y) = x/y.$$

$$f_x(x, y) = \frac{1}{y}$$

$$f(x, y) = xy^{-1} \Rightarrow f_y(x, y) = x(-1)y^{-2} = -xy^{-2} = -\frac{x}{y^2}$$

2.7.8. Find the partial derivatives $S_a(a, b)$ and $S_b(a, b)$ for

$$S(a, b) = \sum_{i=1}^{10} (a^2 - b + i)^2.$$

$$S_a(a, b) = \sum_{i=1}^{10} 2(a^2 - b + i) \cdot 2a = \sum_{i=1}^{10} 4a(a^2 - b + i)$$

$$S_b(a, b) = \sum_{i=1}^{10} 2(a^2 - b + i) \cdot (-1) = -2 \sum_{i=1}^{10} (a^2 - b + i)$$

2.7.10. Find the critical points, i.e., the points (x_0, y_0) that satisfy both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$ for

$$f(x, y) = 3x^4 + 5x - 8y^3.$$

$$f_x(x, y) = 12x^3 + 5, \quad f_y(x, y) = -24y^2$$

$$f_x(x, y) = 0 \Rightarrow 12x^3 + 5 = 0 \Rightarrow x^3 = -\frac{5}{12} \Rightarrow x = -\sqrt[3]{\frac{5}{12}}$$

$$f_y(x, y) = 0 \Rightarrow -24y^2 = 0 \Rightarrow y = 0$$

So the critical point is

$$\left(-\sqrt[3]{\frac{5}{12}}, 0 \right).$$

2.7.24. Use the given data

$$T_0 = 50, T_1 = 20, T_2 = 12, T_3 = 8, T_4 = 5, T_5 = 4$$

to construct the best approximate linear iterative model.

The points we would like to use to make linear model are (T_i, T_{i+1}), which are

$$(50, 20), (20, 12), (12, 8), (8, 5), (5, 4).$$

$$\sum_{i=1}^{5} x_i^2 = 50^2 + 20^2 + 12^2 + 8^2 + 5^2 = 3133$$

$$\sum_{i=1}^{5} x_i = 50 + 20 + 12 + 8 + 5 = 95$$

$$\sum_{i=1}^{5} y_i = 20 + 12 + 8 + 5 + 4 = 49$$

$$\sum_{i=1}^{5} x_i y_i = 50 \cdot 20 + 20 \cdot 12 + 12 \cdot 8 + 8 \cdot 5 + 5 \cdot 4 = 1396$$

Two equations that $y = ax + b$ minimizing the total error satisfies are

$$\left(\sum_{i=1}^{5} x_i^2\right) a + \left(\sum_{i=1}^{5} x_i\right) b = \sum_{i=1}^{5} x_i y_i,$$

$$\left(\sum_{i=1}^{5} x_i\right) a + 5b = \sum_{i=1}^{5} y_i.$$

Therefore

$$\begin{aligned} 3133a + 95b &= 1396 \\ 95b + 5b &= 49. \end{aligned}$$

If you solve this system of linear equations, we get $a \approx 0.35, b \approx 3.15$.

Therefore the least square line we are looking for is $y = 0.35x + 3.15$ and the linear iterative model is

$$T_{n+1} = 0.35T_n + 3.15.$$