Homework 6 Solution

Section 3.1.

3.1.4. Find the logistic model that satisfies:

(a) $C = 20,000, P_0 = 5000, P_1 = 3000.$

$$P_{n+1} = rP_n\left(1 - \frac{P_n}{20000}\right)$$

$$3000 = P_1 = rP_0\left(1 - \frac{P_0}{20000}\right) = r \cdot 5000\left(1 - \frac{5000}{20000}\right) = 3750r$$

$$\Rightarrow r = \frac{3000}{3750} = \frac{4}{5}$$

$$P_{n+1} = \frac{4}{5}P_n\left(1 - \frac{P_n}{20000}\right)$$

(b) $P_0 = 5000, P_1 = 8000, P_2 = 6000.$

$$P_1 = rP_0\left(1 - \frac{P_0}{C}\right) \Rightarrow 8000 = 5000r\left(1 - \frac{5000}{C}\right) \Rightarrow r - \frac{5000r}{C} = \frac{8000}{5000} = \frac{8}{5}$$

$$P_2 = rP_1\left(1 - \frac{P_1}{C}\right) \Rightarrow 6000 = 8000r\left(1 - \frac{8000}{C}\right) \Rightarrow r - \frac{8000r}{C} = \frac{6000}{8000} = \frac{6}{8}$$

$$\frac{3000r}{C} = r - \frac{5000r}{C} - \left(r - \frac{8000r}{C}\right) = \frac{8}{5} - \frac{6}{8} = \frac{17}{20}$$

$$\Rightarrow r = \frac{17C}{60000} \Rightarrow \frac{17C}{60000} - \frac{5000 \cdot 17C}{60000C} = \frac{8}{5} \Rightarrow \frac{17C}{60000} = \frac{8}{5} + \frac{17}{12} = \frac{181}{60}$$

$$\Rightarrow C = \frac{181000}{17}, r = \frac{17 \cdot 181000}{60000 \cdot 17} = \frac{181}{60}$$

$$P_{n+1} = \frac{181}{60}P_n\left(1 - \frac{17P_n}{181000}\right)$$

(c) If $P_0 = 5000$ then P_1 is twice P_0, and if $P_0 = 8000$ then P_1 is half P_0.

$$P_0 = 5000 \Rightarrow 10000 = 2P_0 = P_1 = 5000r\left(1 - \frac{5000}{C}\right) \Rightarrow \frac{2}{r} = \left(1 - \frac{5000}{C}\right)$$

$$P_0 = 8000 \Rightarrow 4000 = \frac{P_0}{2} = P_1 = 8000r\left(1 - \frac{8000}{C}\right) \Rightarrow \frac{1}{2r} = \left(1 - \frac{8000}{C}\right)$$

$$\Rightarrow \left(1 - \frac{5000}{C}\right) = 4\left(1 - \frac{8000}{C}\right) \Rightarrow 1 - \frac{5000}{C} = 4 - \frac{32000}{C}$$

$$\Rightarrow \frac{27000}{C} = 3 \Rightarrow C = 9000$$

$$\frac{2}{r} = 1 - \frac{5000}{9000} = \frac{4000}{9000} \Rightarrow r = \frac{18000}{4000} = \frac{9}{2}$$

$$P_{n+1} = \frac{9}{2}P_n\left(1 - \frac{P_n}{9000}\right)$$

(d) The maximum population $P_1 = 45,000$ occurs when $P_0 = 35,000$.
For $f(x) = rx\left(1 - \frac{x}{C}\right)$, the maximum occurs when $x = \frac{C}{2}$ and the maximum is $f(\frac{C}{2}) = r\frac{C}{2}\left(1 - \frac{C}{2C}\right) = \frac{rC}{4}$. So $\frac{C}{2} = 35000$ and $\frac{rC}{4} = 45000$.

$$\Rightarrow C = 70000, r = \frac{4 \cdot 45000}{C} = \frac{18}{7}$$

$$P_{n+1} = \frac{18}{7}P_n\left(1 - \frac{P_n}{70000}\right)$$

3.1.8. Find the nonlinear population model II $P_{n+1} = rP_n e^{-P_n/N}$ that satisfies:

(a) $N = 1000, P_0 = 100, P_1 = 250$

$$250 = P_1 = rP_0e^{-\frac{P_0}{1000}} = 100re^{-\frac{100}{1000}} = 100re^{-\frac{1}{10}}$$

$$\Rightarrow r = \frac{250}{100}e^{\frac{1}{10}} \approx 2.7629$$

$$P_{n+1} = 2.7629P_ne^{-\frac{P_n}{1000}}$$

(b) $r = 5, P_0 = 100, P_1 = 250$

$$250 = P_1 = 5P_0e^{-\frac{P_0}{N}} = 500e^{-\frac{100}{N}} \Rightarrow e^{-\frac{100}{N}} = \frac{1}{2}$$

$$\Rightarrow -\frac{100}{N} = \ln\frac{1}{2} = -\ln 2 \Rightarrow N = \frac{100}{\ln 2} \approx 144.2695$$

$$P_{n+1} = 5P_ne^{-\frac{P_n}{144.2695}}$$

(c) $P_0 = 2000, P_1 = 6000, P_2 = 4000$

$$P_1 = rP_0e^{-\frac{P_0}{N}} \Rightarrow 6000 = 2000re^{-\frac{2000}{N}} \Rightarrow r = 3e^{\frac{2000}{N}}$$

$$P_2 = rP_1e^{-\frac{P_1}{N}} \Rightarrow 4000 = 6000re^{-\frac{6000}{N}} \Rightarrow r = \frac{2}{3}e^{\frac{6000}{N}}$$

$$\Rightarrow 3e^{\frac{2000}{N}} = \frac{2}{3}e^{\frac{6000}{N}} \Rightarrow \frac{9}{2} = e^{\frac{6000}{N} - \frac{2000}{N}} = e^{\frac{4000}{N}}$$

$$\Rightarrow \frac{4000}{N} = \ln\frac{9}{2} \Rightarrow N = \frac{4000}{\ln\frac{9}{2}} \approx 2659.4376$$

$$r = 3e^{\frac{2000}{\frac{4000}{\ln\frac{9}{2}}}} = 3e^{\frac{\ln\frac{9}{2}}{2}} \approx 6.3640$$

$$P_{n+1} = 6.3640 P_n e^{-\frac{P_n}{2659.4376}}$$

(d) The maximum population $P_1 = 7500$ occurs when $P_0 = 5000$.
For the function $f(x) = rxe^{-\frac{x}{N}}$,

$$f'(x) = re^{-\frac{x}{N}} + rxe^{-\frac{x}{N}} \cdot \left(-\frac{1}{N}\right) = re^{-\frac{x}{N}}\left(1 - \frac{x}{N}\right)$$

$$f'(x) \Leftrightarrow 1 - \frac{x}{N} = 0 \Leftrightarrow x = N$$

Furthermore, if $x < N$ then $f'(x) > 0$ if $x > N$, then $f'(x) < 0$. So $x = N$ is a local maximum. Therefore the maximum population occurs when $x = N$ and it is $f(N)$. So $N = 5000$ and

$$7500 = 5000re^{-\frac{5000}{5000}} = 5000re^{-1} \Rightarrow r = \frac{7500}{5000}e = 1.5e.$$

$$P_{n+1} = 1.5eP_n e^{-\frac{P_n}{5000}}$$

3.1.10. Identify r, s and N for each of the following nonlinear infection model $I_{n+1} = I_n - rI_n + sI_n(1 - I_n/N)$:

(a) $I_{n+1} = 0.7I_n + 2.9I_n(1 - I_n/7500)$

$$I_{n+1} = I_n - 0.3I_n + 2.9I_n(1 - \frac{I_n}{7500})$$

$r = 0.3, s = 2.9, N = 7500$

(b) $I_{n+1} = 3.25I_n(1 - 0.0025I_n)$

$$I_{n+1} = I_n - I_n + 3.25I_n\left(1 - \frac{I_n}{\frac{1}{0.0025}}\right) = I_n - I_n + 3.25I_n\left(1 - \frac{I_n}{400}\right)$$

$r = 1, s = 3.25, N = 400$

(c) $I_{n+1} = 10I_n - \frac{I_n^2}{100}$ in a population of size 1000

$$I_{n+1} = 10I_n - 10\frac{I_n^2}{1000} = 10I_n\left(1 - \frac{I_n}{1000}\right) = I_n - I_n + 10I_n\left(1 - \frac{I_n}{1000}\right)$$

$r = 1, s = 10, N = 1000$

(d) $I_{n+1} = 5.75I_n - \dfrac{I_n^2}{1000}$ in a population of size 5000.

$$I_{n+1} = 5.75I_n - 5\frac{I_n^2}{5000} = 0.75I_n + 5I_n - 5\frac{I_n^2}{5000} = I_n - 0.25I_n + 5I_n\left(1 - \frac{I_n}{5000}\right)$$

$r = 0.25, s = 5, N = 5000$

3.1.16. Find the nonlinear price model $P_{n+1} = \dfrac{a}{P_n} + bP_n + c$ that satisfies each of the following:

(a) $c = 0, P_0 = 1, P_1 = 4, P_2 = 2$

$$P_{n+1} = \frac{a}{P_n} + bP_n \Rightarrow 4 = P_1 = \frac{a}{P_0} + bP_0 = a + b$$

$$2 = P_2 = \frac{a}{P_1} + bP_1 = \frac{a}{4} + 4b$$

$$\Rightarrow a + b = 4, \frac{a}{4} + 4b = 2 \Rightarrow b = \frac{4}{15}, a = \frac{56}{15}$$

$$P_{n+1} = \frac{56}{15P_n} + \frac{4}{15}P_n$$

(b) $c = 2$, and $P_0 = 3.5$ gives the minimum price $P_1 = 2.5$

$$P_{n+1} = \frac{a}{P_n} + bP_n + 2$$

For $f(x) = \dfrac{a}{x} + bx + 2$,

$$f'(x) = -\frac{a}{x^2} + b \Rightarrow f'(x) = 0 \Leftrightarrow x = \pm\sqrt{\frac{b}{a}}$$

For a positive x, $f(x)$ has the minimum if $x = \sqrt{\dfrac{b}{a}}$ and the minimum is $f(x)$. So

$$\sqrt{\frac{b}{a}} = 3.5, -\frac{a}{\sqrt{\frac{b}{a}}} + b\sqrt{\frac{b}{a}} + 2 = 2.5.$$

$$\Rightarrow \frac{b}{a} = 3.5^2 = 12.25 \Rightarrow b = 12.25a$$

$$-\frac{a}{3.5} + 3.5b + 2 = 2.5 \Rightarrow -\frac{a}{3.5} + 42.875a + 2 = 2.5 \Rightarrow a \approx 0.0017, b \approx 0.1438$$

$$P_{n+1} = \frac{0.0017}{P_n} + 0.1438P_n + 2$$