Homework 7 Solution

Section 3.2.

3.2.10. For the equation $x_{n+1} = bx_n^a$ where a and b are positive constants:

(a) Compute $x_1, \cdots, x_5$, in terms of x_0.

$$x_1 = bx_0^a$$

$$x_2 = bx_1^a = b(bx_0^a)^a = b \cdot b^a (x_0^a)^a = b^{1+a} x_0^{a^2}$$
$$x_3 = bx_2^a = b(b^{1+a} x_0^{a^2})^a = b \cdot b^{(1+a)a} x_0^{a^2 \cdot a} = b^{1+a+a^2} x_0^{a^3}$$
$$x_4 = bx_3^a = b(b^{1+a+a^2} x_0^{a^3})^a = b \cdot b^{(1+a+a^2)a} x_0^{a^3 \cdot a} = b^{1+a+a^2+a^3} x_0^{a^4}$$
$$x_5 = bx_4^a = b(b^{1+a+a^2+a^3} x_0^{a^4})^a = b \cdot b^{(1+a+a^2+a^3)a} x_0^{a^4 \cdot a} = b^{1+a+a^2+a^3+a^4} x_0^{a^5}$$

(b) Can you construct a formula for the exact solution x_n for all n?
From the observation above, we can conclude that

$$x_n = b^{1+a+a^2+\cdots+a^{n-1}} x_0^{a^n}.$$

(c) Use the geometric series to simplify your solution in part (b).

$$x_n = b^{1+a+a^2+\cdots+a^{n-1}} x_0^{a^n} = b^{\sum_{i=0}^{n-1} a^i} x_0^{a^n}$$

3.2.12. The formula for the exact solution of $x_{n+1} = (x_n - c)^a + c$, where c is a constant, can be derived as follows:

(a) Use the substitution $u_n = x_n - c$ to convert $x_{n+1} = (x_n - c)^a + c$ into $u_{n+1} = u_n^a$.

$$x_{n+1} = (x_n - c)^a + c = u_n^a + c \Rightarrow u_{n+1} = x_{n+1} - c = u_n^a$$

(b) Write the solution for u_n for all n.

$$u_n = u_0^{a^n}$$

(c) Use the substitution $u_n = x_n - c$ again to find a formula for x_n for all n.

$$x_n - c = (x_0 - c)^{a^n} \Rightarrow x_n = (x_0 - c)^{a^n} + c$$

3.2.14. Find all fixed points (if any) of $f(x) = 4x^5$.

$$4x^5 = x \Rightarrow 4x^5 - x = 0 \Rightarrow x(4x^4 - 1) = 0 \Rightarrow x((2x^2)^2 - 1) = 0$$

$$\Rightarrow x(2x^2 - 1)(2x^2 + 1) = 0 \Rightarrow x = 0, 2x^2 = 1$$

$$\Rightarrow x = 0, \pm\frac{1}{\sqrt{2}}$$

3.2.18. Find all fixed points (if any) of $P_{n+1} = 4P_n e^{-P_n}$.

$$f(x) = 4xe^{-x}$$

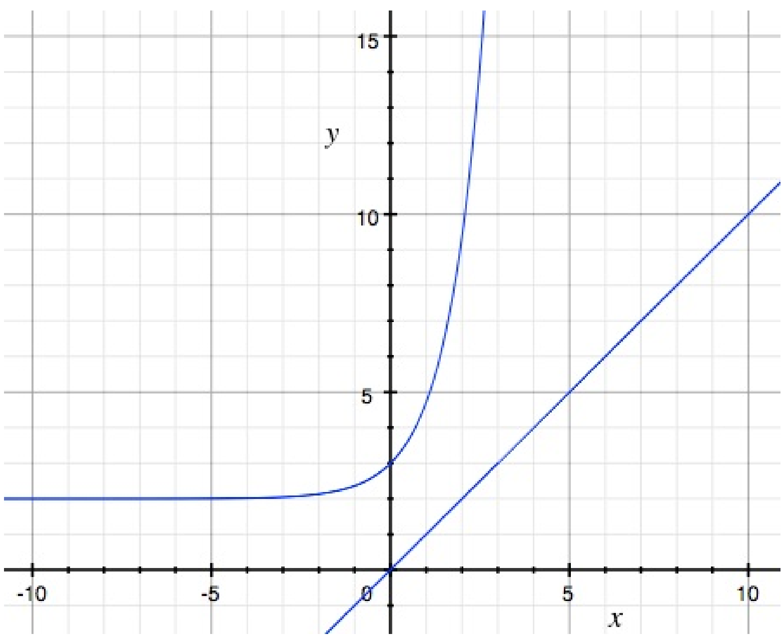
$$f(x) = x \Rightarrow 4xe^{-x} = x \Rightarrow 4xe^{-x} - x = 0$$

$$\Rightarrow x(4e^{-x} - 1) = 0 \Rightarrow x = 0, 4e^{-x} = 1$$

$$4e^{-x} = 1 \Rightarrow e^x = 4 \Rightarrow x = \ln 4$$

$$x = 0, \ln 4$$

3.2.22. A fixed point of $f(x)$ may be viewed as a point where the graphs of $y = f(x)$ and $y = x$ intersect. Determine graphically whether $f(x) = e^x + 2$ have fixed points and how many.



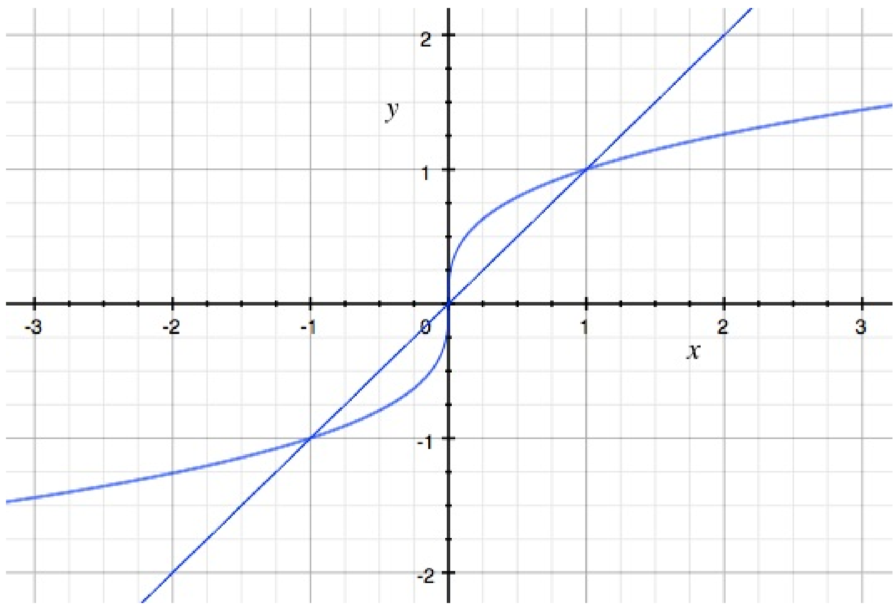
There is no intersection point of two graphs. So there is no fixed point.

3.2.26. Find all fixed points of $x_{n+1} = x_n^{1/3}$ and determine their stability using the formula for the exact solution $x_n = x_0^{a^n}$. Find the basins of attraction for those that are stable.

$$x = x^{\frac{1}{3}} \Rightarrow x^3 = x \to x^3 - x = 0 \Rightarrow x(x-1)(x+1) = 0$$
$$\Rightarrow x = 0, -1, 1$$

Thus there are three fixed points.

The graph of $y = x^{1/3}$ and $y = x$ are:



From the graph, we can observe that if $0 < x_n < 1$, then $x_{n+1} = x_n^{1/3}$ is between 0 and 1 and $x_{n+1} > x_n$. In particular, for any x_0 with $0 < x_0 < 1$, $|x_0 - 0| = x_0 < x_1 = |x_1 - 0|$ and 0 is unstable. By the same reason,

$$|x_{n+1} - 1| = 1 - x_{n+1} < 1 - x_n = |x_n - 1|.$$

On the other hand, if $x_n > 1$, then $x_{n+1} = x_n^{1/3} < x_n$ and $x_{n+1} > 1$. Then we have

$$|x_{n+1} - 1| = x_{n+1} - 1 < x_n - 1 = |x_n - 1|.$$

Finally,

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_0^{\left(\frac{1}{3}\right)^n} = x_0^{\lim_{n\to\infty}\left(\frac{1}{3}\right)^n} = x_0^0 = 1.$$

Therefore 1 is locally stable, and its basin of attraction is $(0, \infty)$.

By a similar argument, one can show that -1 is also a locally stable point with a basin of attraction $(-\infty, 0)$.