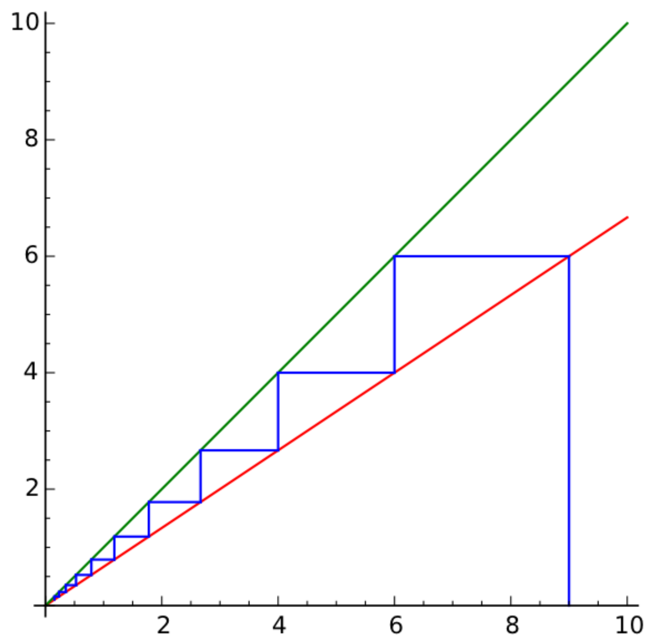
Homework 8 Solution

Section 3.3.

3.3.4. For $f(x) = 2x/3$, draw the first few steps of a cobweb graph starting at $x_0 = 9$.

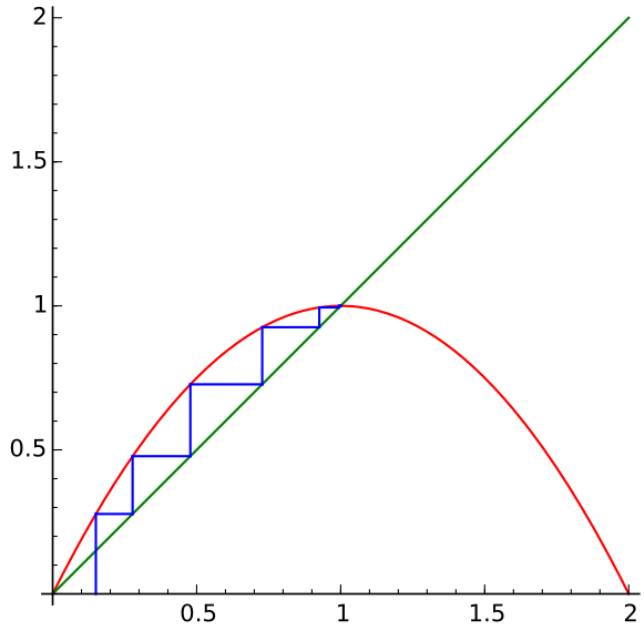


3.3.10. Find all fixed points of $I_{n+1} = 2I_n - I_n^2$ and use cobwebbing to determine their stability and any oscillation around them.

Let $f(x) = 2x - x^2$.

$$f(x) = x \Rightarrow 2x - x^2 = x \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0$$

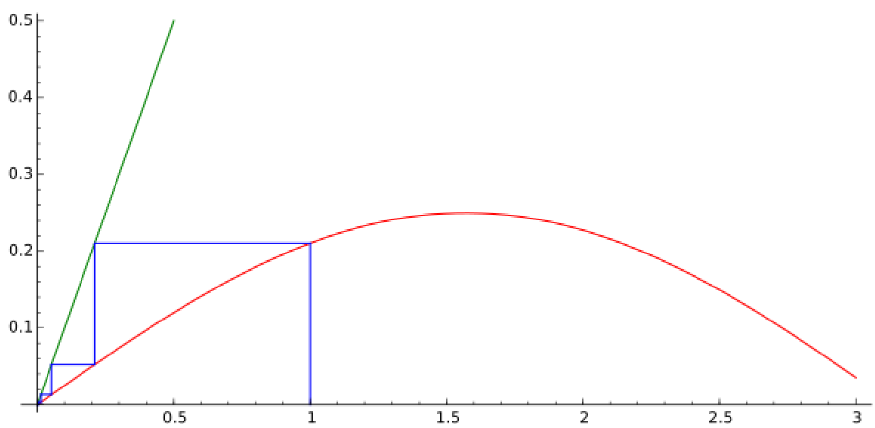
$$\Rightarrow x = 0, x = 1$$



The cobweb graph shows that 0 is a repeller and 1 is a attractor. So 0 is unstable and 1 is stable. Also it shows that there is no oscillation.

3.3.12. Find all fixed points of $x_{n+1} = 0.25 \sin x_n$ and use cobwebbing to determine their stability and any oscillation around them.

Note that $x = 0$ is a fixed point of $f(x) = 0.25 \sin x$. We claim that it is a unique fixed point. If we define $g(x) = 0.25 \sin x - x$, then $g(0) = 0$. Moreover, $g'(x) = 0.25 \cos x - 1 < 0.25 - 1 < 0$, so it is a strictly decreasing function. Therefore 0 is the only zero of $g(x)$, and equivalently, 0 is the only fixed point of $f(x)$.



From the cobweb graph, we can conclude that 0 is stable and there is no oscillation.

3.3.14. Use the derivative to determine the stability of and any oscillation around the given fixed point $p = 1$ for

$$f(x) = (x^2 - 3x + 5)/3.$$

$$f'(x) = (2x - 3)/3 = \frac{2}{3}x - 1$$

$$f'(1) = -\frac{1}{3}$$

Because $|f'(1)| = 1/3 < 1$, 1 is a stable fixed point. Also $f'(1) < 1$, so there is an oscillation around it.

3.3.20. Find all fixed points of

$$x_{n+1} = x_n^{1/4}, x_0 > 0$$

and use the derivative to determine their stability and any oscillation around them.

$$f(x) = x^{1/4}$$

$$f(x) = x \Rightarrow x^{1/4} = x \Rightarrow x^4 = x$$

$$\Rightarrow x(x^3 - 1) = 0 \Rightarrow x(x - 1)(x^2 + x + 1) = 0 \Rightarrow x = 0, x = 1$$

But because $x_0 > 0$, we should exclude $x = 0$.

$$f'(x) = \frac{1}{4}x^{-\frac{3}{4}}$$

$$f'(1) = \frac{1}{4} \Rightarrow |f'(1)| = \frac{1}{4} < 1$$

So 1 is a stable fixed point. Because the derivative is positive, there is no oscillation around them.

3.3.22. Find all non-negative fixed points of $P_{n+1} = rP_n e^{-P_n/1000}$ and use the derivative to determine their stability.

$$f(x) = rxe^{-x/1000}$$

$$f(x) = x \Rightarrow rxe^{-x/1000} = x \Rightarrow x(re^{-x/1000} - 1) = 0$$

$$\Rightarrow e^{-x/1000} = 1/r \Rightarrow -x/1000 = \ln 1/r = -\ln r \Rightarrow x = 1000 \ln r$$

(a) $r = 0.5$

The nonzero fixed point is $1000 \ln r = 1000 \ln 0.5 \approx -693.147$. So there is no positive fixed point.

(b) $r = 5$

The nonnegative fixed point is $1000 \ln r = 1000 \ln 5 \approx 1609.438$.

$$f'(x) = 5e^{-x/1000} - \frac{5}{1000}xe^{-x/1000} = 5\left(1 - \frac{x}{1000}\right)e^{-x/1000}$$

$$f'(1000 \ln 5) = 5(1 - \ln 5)e^{-\ln 5} \approx -0.609$$

Because $|f'(1000 \ln 5) = 0.609 < 1$, it is stable.