Homework 9 Solution

Section 3.5 $\sim$ 3.7.

3.5.8. Find all solutions of $f(f(x)) = x$ for $f(x) = -x^3/2$ and determine which are fixed points of $f(x)$ and which are 2-cycles.

$$f(f(x)) = f(-\frac{x^3}{2}) = -\frac{\left(-\frac{x^3}{2}\right)^3}{2} = \frac{x^9}{16}$$

$$f(f(x)) = x \Rightarrow \frac{x^9}{16} - x = 0 \Rightarrow x^9 - 16x = 0$$

$$\Rightarrow x(x^8 - 16) = 0 \Rightarrow x(x^4 - 4)(x^4 + 4) = 0 \Rightarrow x(x^2 - 2)(x^2 + 2)(x^4 + 4) = 0$$

$$\Rightarrow x(x - \sqrt{2})(x + \sqrt{2})(x^2 + 2)(x^4 + 4) = 0 \Rightarrow x = 0, \sqrt{2}, -\sqrt{2}$$

$$f(0) = 0, f(\sqrt{2}) = -\sqrt{2}, f(-\sqrt{2}) = \sqrt{2}$$

Therefore 0 is a fixed point of $f(x)$, and $\{\sqrt{2}, -\sqrt{2}\}$ is a 2-cycle.

3.5.10. Find all fixed points of $f(x) = -x^2 + 2x + 2$ and then use them to help find all points of period 2.

$$f(x) = x \Rightarrow -x^2 + 2x + 2 = x \Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow x = \frac{1 \pm \sqrt{9}}{2}$$

$$f(f(x)) = f(-x^2 + 2x + 2) = -(-x^2 + 2x + 2)^2 + 2(-x^2 + 2x + 2) + 2$$

$$= -x^4 + 4x^3 - 2x^2 - 4x + 2$$

$$f(f(x)) = x \Rightarrow -x^4 + 4x^3 - 2x^2 - 4x + 2 = x \Rightarrow x^4 - 4x^3 + 2x^2 + 5x - 2 = 0$$

Then $x^2 - x - 2$ is a factor of $x^4 - 4x^3 + 2x^2 + 5x - 2$, and

$$x^4 - 4x^3 + 2x^2 + 5x - 2 = (x^2 - x - 2)(x^2 - 3x + 1)$$

$$x^2 - 3x + 1 = 0 \Rightarrow x = \frac{3 \pm \sqrt{5}}{2}$$

One can check that $f(\frac{3+\sqrt{5}}{2}) = \frac{3-\sqrt{5}}{2}$ and $f(\frac{3-\sqrt{5}}{2}) = \frac{3+\sqrt{5}}{2}$. Therefore

$$\{\frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}\}$$

is a 2-cycle.

3.5.18. Given that $a = 1$ is one point of a 2-cycle of $f(x) = 2 - 2^x$, find the other point b and determine the stability of the cycle.

$$f(1) = 2 - 2^1 = 0,\ f(0) = 2 - 2^0 = 1$$

So $b = 1$.

3.5.22. The equation $P_{n+1} = \dfrac{1}{P_n} + \dfrac{P_n}{2} - 1$ is a price model.

(a) Find the two equilibrium points and determine their stability.

$$f(x) = \frac{1}{x} + \frac{x}{2} - 1$$

$$x = f(x) \Rightarrow x = \frac{1}{x} + \frac{x}{2} - 1 \Rightarrow \frac{x}{2} - \frac{1}{x} + 1 = 0 \Rightarrow x^2 + 2x - 2 = 0$$

$$\Rightarrow x = \frac{-2 \pm \sqrt{12}}{2} = -1 \pm \sqrt{3}$$

$$f'(x) = -\frac{1}{x^2} + \frac{1}{2}$$

$$f'(-1+\sqrt{3}) = -\frac{1}{(-1+\sqrt{3})^2} + \frac{1}{2} \approx -1.3660 < -1$$

So $-1 + \sqrt{3}$ is an unstable equilibrium point.

$$|f'(-1-\sqrt{3})| = |-\frac{1}{(-1-\sqrt{3})^2} + \frac{1}{2}| \approx 0.3660 < 1$$

Therefore $-1 - \sqrt{3}$ is a stable fixed point.

(b) Use the results from part (a) to help find the points of the 2-cycle, and then determine the stability of that cycle.

$$f(x) = \frac{1}{x} + \frac{x}{2} - 1 = \frac{x^2 - 2x + 2}{2x}$$

$$f(f(x)) = f(\frac{x^2 - 2x + 2}{2x}) = \frac{2x}{x^2 - 2x + 2} + \frac{x^2 - 2x + 2}{4x} - 1$$

$$= \frac{x^4 - 8x^3 + 24x^2 - 16x + 4}{4x^3 - 8x^2 + 8x}$$

$$f(f(x)) = x \Rightarrow \frac{x^4 - 8x^3 + 24x^2 - 16x + 4}{4x^3 - 8x^2 + 8x} = x$$

$$\Rightarrow x^4 - 8x^3 + 24x^2 - 16x + 4 = 4x^4 - 8x^3 + 8x^2 \Rightarrow 3x^4 - 16x^2 + 16x - 4 = 0$$

Note that $x^2 + 2x - 2$ is a factor of $3x^4 - 16x^2 + 16x - 4$. Indeed,

$$3x^4 - 16x^2 + 16x - 4 = (x^2 + 2x - 2)(3x^2 - 6x + 2).$$

$$3x^2 - 6x + 2 = 0 \Rightarrow x = \frac{6 \pm \sqrt{12}}{6} = \frac{3 \pm \sqrt{3}}{3}$$

$$|f'(\frac{3+\sqrt{3}}{3})f'(\frac{3-\sqrt{3}}{3})| = 0.5 < 1$$

Therefore the 2-cycle

$$\{\frac{3+\sqrt{3}}{3}, \frac{3-\sqrt{3}}{3}\}$$

is stable.

3.5.30. (a) Suppose $f^4(p) = p$ for some point p and some function $f(x)$. What are the possible periods that p might have?

Because the period is the smallest m so that $f^m(p) = p$, the period of p is at most 4. But 3 is impossible, because if $f^3(p) = p$, then $p = f^4(p) = f(f^3(p)) = f(p)$ and p is a fixed point (so the period is 1). Therefore the possible periods are 1, 2, and 4.

(b) Suppose $f^{12}(p) = p$ for some point p and some function $f(x)$. What are the possible periods that p might have?

By the same reason, the period of p is at most 12. We claim that the period of p must be a divisor of 12, so 1, 2, 3, 4, 6, and 12 are the only possible cases. Indeed, if the period m of p is less than 12, then $f^m(p) = p$. If we divide 12 by m, then $12 = qm + r$ for some $q \geq 1$ and the remainder $m > r \geq 0$. Now $p = f^{12}(p) = f^r(f^{qm}(p)) = f^r(p)$, so the period of p is at most r unless $r = 0$. The first case is impossible because the period m is larger then r. Therefore $r = 0$ and m is a divisor of 12.

3.5.32. For $f(x) = 1 - 4|x|$, $p_1 = -11/65$ is a point of an m-cycle. Find all other points of that cycle, and determine its period and stability.

$$f(-\frac{11}{65}) = 1 - 4|-\frac{11}{65}| = \frac{21}{65}$$

$$f(\frac{21}{65}) = 1 - 4|\frac{21}{65}| = -\frac{19}{65}$$

$$f(-\frac{19}{65}) = 1 - 4|\frac{19}{65}| = -\frac{11}{65}$$

Therefore $p = -11/65$ has period 3. Note that

$$f(x) = \begin{cases} 1 - 4x, & x > 0 \\ 1 + 4x, & x < 0. \end{cases}$$

So

$$f'(x) = \begin{cases} -4, & x > 0 \\ 4, & x < 0 \end{cases}$$

Thus
$$|f'(-\frac{11}{65})f'(\frac{21}{65})f'(-\frac{19}{65})| = 64 > 1$$
and the 3-cycle is unstable.

3.6.2. Convert $P_{n+1} = rP_n(1 - P_n^2/C^2)$ into a one-parameter family.

Let $x_n = P_n/C$. Then
$$Cx_{n+1} = rCx_n(1 - x_n^2) \Rightarrow x_{n+1} = rx_n(1 - x_n^2)$$

3.6.8. Find all positive fixed points of $f_c(x) = \dfrac{cx^2}{x^2+1}$ and their intervals of existence for $c > 0$.
$$f_c(x) = x \Rightarrow \frac{cx^2}{x^2+1} = x \Rightarrow cx^2 = x^3 + x \Rightarrow x^3 - cx^2 + x = 0$$
$$\Rightarrow x(x^2 - cx + 1) = 0$$
$$\Rightarrow x = 0, x = \frac{c \pm \sqrt{c^2-4}}{2}$$
Because $c > 0$, $\dfrac{c+\sqrt{c^2-4}}{2} > 0$. Furthermore, $c^2 > c^2 - 4$ implies $c > \sqrt{c^2-4}$. So $\dfrac{c-\sqrt{c^2-4}}{2} > 0$ as well. But these real solution exist only if $c^2 - 4 \geq 0$ or equivalently, $c \geq 2$ (note that $c > 0$). Therefore the interval of existence is $[2, \infty)$.

3.6.12. Find the interval of stability of the fixed point 0 for $r > 0$ of $f_r(x) = rx^2(1-x)$.
$$f_r'(x) = 2rx - 3rx^2 \Rightarrow |f_r'(0)| = 0 < 1.$$
Therefore the fixed point 0 is always stable, so the interval of stability is $(0, \infty)$.

3.6.22. Find the 2-cycles and their intervals of existence and stability for $a > 0$ of $f_a(x) = -ax^3$.
$$f_a(x) = x \Rightarrow -ax^3 = x \Rightarrow x(-ax^2 - 1) = 0 \Rightarrow x = 0$$
(Note that $-ax^2 - 1 < 0$.) So $x = 0$ is the unique fixed point of $f_a(x)$.
$$f_a(f_a(x)) = f_a(-ax^3) = -a(-ax^3)^3 = a^4x^9$$
$$f_a(f_a(x)) = x \Rightarrow a^4x^9 = x \Rightarrow x(a^4x^8 - 1) = 0 \Rightarrow x(ax^2-1)(ax^2+1)(a^2x^4+1) = 0$$
$$\Rightarrow x = 0, x = \frac{1}{\sqrt{a}}, x = -\frac{1}{\sqrt{a}}$$

Since $f_a(\frac{1}{\sqrt{a}}) = -\frac{1}{\sqrt{a}}$ and $f_a(-\frac{1}{\sqrt{a}}) = \frac{1}{\sqrt{a}}$, $\{\frac{1}{\sqrt{a}}, -\frac{1}{\sqrt{a}}\}$ is a 2-cycle. It exists for every $a > 0$. Therefore the interval of existence is $(0, \infty)$.

$$f_a'(x) = -3ax^2$$

$$|f_a'(\frac{1}{\sqrt{a}})f_a'(-\frac{1}{\sqrt{a}})| = 9 > 1$$

Therefore the 2-cycle is always unstable and the interval of stability is an emptyset.

3.6.28. For the threshold population model $P_{n+1} = \frac{rP_n^2}{C}(1-P_n/C)$, when r is small the population will become extinct regardless of the initial population size P_0. When r is larger, although small initial populations still lead to extinction (because of the threshold), there is a positive equilibrium population.

(a) Find the smallest growth rate r for which a positive equilibrium population exists.

Set $x_n = P_n/C$ or equivalently, $P_n = Cx_n$. Then

$$Cx_{n+1} = r\frac{(Cx_n)^2}{C}(1 - x_n) \Rightarrow x_{n+1} = rx_n^2(1 - x_n)$$

Let $f_r(x) = rx^2(1 - x) = rx^2 - rx^3$.

$$f_r(x) = x \Rightarrow rx^2 - rx^3 = x \Rightarrow rx^3 - rx^2 + x = 0 \Rightarrow x(rx^2 - rx + 1) = 0$$

$$\Rightarrow x = 0, x = \frac{r \pm \sqrt{r^2 - 4r}}{2r}$$

Note that $r^2 > r^2 - 4r$, so $r > \sqrt{r^2 - 4r}$ and two zeros are all positive real numbers if $r^2 - 4r \geq 0$ or equivalently $r \geq 4$. Therefore the smallest r with positive fixed population is $r = 4$.

(b) Find the interval of r-values for which there exists a stable positive equilibrium population.

$$f_r'(x) = 2rx - 3rx^2 = 3(rx - rx^2) - rx$$

Note that for two positive fixed points, $rx - rx^2 = 1$ because they are zeros of $rx^2 - rx + 1 = 0$.

$$f_r'(\frac{r + \sqrt{r^2 - 4r}}{r}) = 3 - r - \sqrt{r^2 - 4r}$$

$$|f_r'(\frac{r + \sqrt{r^2 - 4r}}{r})| < 1 \Leftrightarrow -1 < 3 - r - \sqrt{r^2 - 4r} < 1$$

$$\Leftrightarrow 2 < r + \sqrt{r^2 - 4r} < 4$$

But because $r \ge 4$, it is impossible. In other words, the fixed point $\dfrac{r+\sqrt{r^2-4r}}{r}$ is never stable.

On the other hand,

$$f_r'(\frac{r-\sqrt{r^2-4r}}{r}) = 3 - r + \sqrt{r^2-4r}$$

$$|f_r'(\frac{r-\sqrt{r^2-4r}}{r})| < 1 \Leftrightarrow -1 < 3 - r + \sqrt{r^2-4r} < 1$$

$$\Leftrightarrow 2 < r - \sqrt{r^2-4r} < 4.$$

Note that $(r-2)^2 = r^2 - 4r + 4 > r^2 - 4r$, so $r - 2 > \sqrt{r^2-4r}$ and $r - \sqrt{r^2-4r} > 2$. And if $r > 4$,

$$(r-4)^4 < r(r-4) = r^2 - 4r \Rightarrow r - 4 < \sqrt{r^2-4r} \Rightarrow r - \sqrt{r^2-4r} < 4.$$

Therefore the interval of stability (of the fixed point $\dfrac{r-\sqrt{r^2-4r}}{r}$ is $(4, \infty)$.

3.7.2. Let $f_r(x) = \dfrac{rx}{x+5}$.

(a) Find the interval of stability of the fixed point 0;

$$f_r'(x) = \frac{r(x+5) - rx}{(x+5)^2} = \frac{5r}{(x+5)^2}$$

$$f_r'(0) = \frac{r}{5}$$

$$|f_r'(0)| = |\frac{r}{5}| < \Leftrightarrow r < 5$$

The interval of stability is $(0, 5)$.

(b) Find the positive fixed point $p(r)$ and its intervals of existence and stability;

$$f_r(x) = x \Rightarrow \frac{rx}{x+5} = x \Rightarrow rx = x(x+5)$$

$$\Rightarrow x^2 + 5x - rx = 0 \Rightarrow x(x+5-r) = 0 \Rightarrow x = 0, x = r - 5$$

The nonzero fixed point $p(r) = r - 5$ is positive when $r > 5$. So the interval of existence is $(5, \infty)$.

$$f_r'(r-5) = \frac{5r}{(r-5+5)^2} = \frac{5}{r}$$

$$|f_r'(r-5)| = \frac{5}{r} < 1 \Leftrightarrow r > 5$$

Therefore the interval of stability is $(5, \infty)$.

(c) Show that $p(r)$ bifurcates from 0 at some parameter value $r = r_0$.
When $r = 5$, $p(5) = 5 - 5 = 0$. At $r = 5$, the stable fixed point 0 becomes unstable and a new positive stable fixed point $p(r)$ bifurcates from 0.

3.7.12. Let $f_a(x) = 0.25 - ax^2$.

(a) Find the non-negative fixed point $p(a)$ and its intervals of existence and stability;

$$f_a(x) = x \Rightarrow 0.25 - ax^2 = x \Rightarrow ax^2 + x - \frac{1}{4} = 0$$

$$\Rightarrow x = \frac{-1 \pm \sqrt{1+a}}{2a}$$

The positive solution is $p(a) = \dfrac{-1+\sqrt{1+a}}{2a}$ and it exists for every $a > 0$. Therefore the interval of existence is $(0, \infty)$.

$$f_a'(x) = -2ax \Rightarrow f_a'(\frac{-1+\sqrt{1+a}}{2a}) = 1 - \sqrt{1+a}$$

So $|f_a'(p(a))| = |1 - \sqrt{1+a}| < 1$ only if $a < 3$. Thus the interval of stability is $(0, 3)$.

(b) Find the 2-cycle $p_1(a), p_2(a)$ and its interval of existence and stability;

$$f_a(f_a(x)) = f_a(0.25 - ax^2) = 0.25 - a(0.25 - ax^2)^2 = \frac{1}{4} - a(\frac{1}{4} - ax^2)^2$$

$$= \frac{1}{4} - \frac{a}{16} + \frac{a^2}{2}x^2 - a^3x^4$$

$$f_a(f_a(x)) = x \Rightarrow a^3x^4 - \frac{a^2}{2}x^2 + x + \frac{a-4}{16} = 0$$

Note that $ax^2 + x - \frac{1}{4}$ is a factor of $a^3x^4 - \frac{a^2}{2}x^2 + x + \frac{a-4}{16}$. Indeed,

$$a^3x^4 - \frac{a^2}{2}x^2 + x + \frac{a-4}{16} = (ax^2 + x - \frac{1}{4})(a^2x^2 - ax + \frac{4-a}{4}).$$

Now $a^2x^2 - ax + \frac{4-a}{4} = 0$ has zeros

$$x = \frac{a \pm \sqrt{a^3 - 3a^2}}{2a^2} = \frac{1 \pm \sqrt{a-3}}{2a}.$$

$$p_1(a) = \frac{1+\sqrt{a-3}}{2a}, \quad p_2(a) = \frac{1-\sqrt{a-3}}{2a}$$

So they are two positive real roots when $a > 3$ and $1 - \sqrt{a-3} > 0$, or equivalently, $a < 4$. Therefore the interval of existence (of a positive 2-cycle) is $(3, 4)$. Of course, the interval of existence (of a 2-cycle) is $(3, \infty)$.

$$|f_a'(\frac{1+\sqrt{a-3}}{2a})f_a'(\frac{1-\sqrt{a-3}}{2a})|$$

$$= |(1+\sqrt{a-3})(1-\sqrt{a-3})| = |1-(a-3)| = |4-a|$$

So it is less than one if $3 < a < 5$. Therefore the interval of stability (of a 2-cycle) is $(3, 5)$, and that of a positive 2-cycle is $(3, 4)$.

(c) Show that the 2-cycle bifurcates from $p(a)$ at some parameter value $a = a_0$.

When $a = 3$, $p(3) = p_1(3) = p_2(3) = \frac{1}{6}$. At $a = 3$, the stable fixed point $p(3)$ becomes unstable and a stable positive 2-cycle $p_1(a), p_2(a)$ bifurcates from $p(a)$.

3.7.16. Sketch the graph of $p(a)$, $p_1(a)$, $p_2(a)$ from Exercise 12 together.