MATH 1700 MIDTERM EXAM 1 SOLUTION

SPRING 2015 - MOON

- Write your answer neatly and show steps.
- Except calculators, any electronic devices including laptops and cell phones are not allowed.
- Do not use the graphing function on your calculator.

(1) Quick survey.

(a) (1 pt) This class is:

Too easy			Moderate			Too difficult
1	2	3	4	5	6	7

(b) (2 pts) Write any suggestion for improving this class. (For instance, give more examples in class, explain proofs of formulas in detail, give more homework, slow down the tempo, ...)

(2) (4 pts) Write the sum

$$\frac{2}{2}+\frac{4}{3}+\frac{8}{4}+\frac{16}{5}+\cdots+\frac{2048}{12}$$

by using the sigma notation.

$$\frac{2}{2}+\frac{4}{3}+\frac{8}{4}+\frac{16}{5}+\cdots+\frac{2048}{12}$$
$$=\frac{2^1}{1+1}+\frac{2^2}{2+1}+\frac{2^3}{3+1}+\frac{2^4}{4+1}+\cdots+\frac{2^{11}}{11+1}=\sum_{k=1}^{11}\frac{2^k}{k+1}$$

- If the index does not match with variable in summands, -1 pt.
- The range of the index is wrong: -1 pt.

Date: February 23, 2014.

(3) Let x_n is the solution of the following discrete dynamical system given by

$$x_0 = 100,\ x_1 = \frac{1}{2}x_0,\ x_2 = \frac{2}{3}x_1 - 10,\ x_3 = \frac{3}{4}x_2 + 20,\ x_4 = \frac{4}{5}x_3 - 30,\ x_5 = \frac{5}{6}x_4 + 40, \cdots .$$

(a) (4 pts) Find a general iterative formula for x_n.

$$x_{n+1} = \frac{n+1}{n+2}x_n + (-1)^n 10n, \quad x_0 = 100$$

(b) (3 pts) Calculate x_5 and x_6.

$$x_1 = \frac{1}{2}x_0 = \frac{1}{2} \cdot 100 = 50$$

$$x_2 = \frac{2}{3}x_1 - 10 = \frac{2}{3} \cdot 50 - 10 = \frac{70}{3}$$

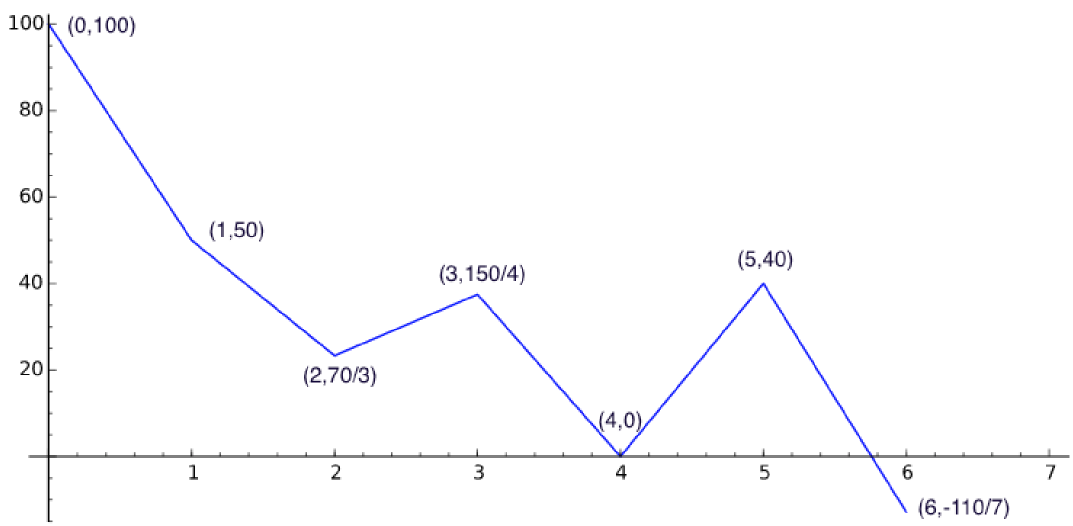
$$x_3 = \frac{3}{4}x_2 + 20 = \frac{3}{4} \cdot \frac{70}{3} + 20 = \frac{150}{4}$$

$$x_4 = \frac{4}{5}x_3 - 30 = \frac{4}{5} \cdot \frac{150}{4} - 30 = 0$$

$$x_5 = \frac{5}{6}x_4 + 40 = \frac{5}{6} \cdot 0 + 40 = 40$$

$$x_6 = \frac{6}{7}x_5 - 50 = \frac{6}{7} \cdot 40 - 50 = -\frac{110}{7}$$

(c) (4 pts) Sketch the time series graph of x_n for $0 \le n \le 6$. Indicate the coordinates of each vertex.



(4) (a) (4 pts) Find a closed simple formula of x_n for the iterative system

$$x_{n+1} = \frac{(n+2)^2}{(n+1)^2} x_n, \quad x_0 = 2.$$

$$x_n = \frac{(n+1)^2}{n^2} x_{n-1} = \frac{(n+1)^2}{n^2} \cdot \frac{n^2}{(n-1)^2} x_{n-2} = \frac{(n+1)^2}{n^2} \cdot \frac{n^2}{(n-1)^2} \cdot \frac{(n-1)^2}{(n-2)^2} x_{n-3} = \cdots$$

$$= \frac{(n+1)^2}{n^2} \cdot \frac{n^2}{(n-1)^2} \cdot \frac{(n-1)^2}{(n-2)^2} \cdots \frac{3^2}{2^2} \cdot \frac{2^2}{1^2} x_0 = \frac{(n+1)^2}{1^2} x_0 = 2(n+1)^2$$

Therefore $x_n = 2(n+1)^2$.

(b) (2 pts) Find x_{2015}.

$$x_{2015} = 2(2015+1)^2 = 8128512$$

(5) (5 pts) Evaluate the following infinite sum

$$6 + \frac{6}{7} + \frac{6}{7^2} + \frac{6}{7^3} + \cdots .$$

$$6 + \frac{6}{7} + \frac{6}{7^2} + \frac{6}{7^3} + \cdots = 6\left(1 + \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \cdots\right)$$

$$= 6\left(1 + \frac{1}{7} + \left(\frac{1}{7}\right)^2 + \left(\frac{1}{7}\right)^3 + \cdots\right) = 6 \cdot \frac{1}{1 - \frac{1}{7}} = \frac{6}{\frac{6}{7}} = 7$$

(6) Suppose that you borrowed \$15,000 at an annual interest rate 12% compounded monthly. You want to make a fixed amount of monthly payment to pay it back.

(a) (4 pts) If your monthly payment is \$200, how much is the remaining balance after 3 years?

By Loan Payment Model $P_{n+1} = (1+i)P_n - d$ with $i = 0.12/12 = 0.01$, $d = 200$, the remaining balance after 3 years (= 36 months) is

$$P_{36} = 15000(1+0.01)^{36} - 200\frac{(1+0.01)^{36}-1}{0.01} \approx 12846.16$$

Therefore the remaining balance is approximately \$12846.16.

(b) (5 pts) If you want to pay back everything in 5 years, how much should be the monthly payment?

We want that $P_{5\cdot 12} = P_{60} = 0$.

$$0 = P_{60} = 15000(1+0.01)^{60} - d\frac{(1+0.01)^{36}-1}{0.01}$$

$$\Rightarrow d\frac{(1+0.01)^{36}-1}{0.01} = 15000(1+0.01)^{60}$$

$$\Rightarrow d = 15000 \cdot 0.01 \cdot \frac{(1+0.01)^{60}}{(1+0.01)^{36}-1} \approx 333.67$$

Thus the monthly payment should be approximately \$333.67.

- Setting up the equation $0 = 15000(1+0.01)^{60} - d\frac{(1+0.01)^{36}-1}{0.01}$: 3 pts.
- Getting the answer \$333.67: 5 pts.

(c) (2 pts) In Item (b), what is the total interest you paid during the 5 years?

$$\begin{aligned}\text{Total interest} &= \text{actual payment} - \text{payed balance} \\ &= 333.67 \cdot 60 - 15000 \approx 5020.20\end{aligned}$$

Thus the total interest is \$5020.20.

(7) It is known that the population growth ratio (in a year) of wild bison in Montana is 1.013. At the end of year, 200 bisons are hunted legally.

(a) (4 pts) Find an iterative model describing the population P_n of bisons in n-th year.

$$P_{n+1} = 1.013P_n - 200$$

(b) (3 pts) Find the fixed point of the population P_n.

$a = 1.013$ and $b = -200$.

$$\text{fixed point } = \frac{b}{1-a} = \frac{-200}{1-1.013} \approx 15385$$

(c) (2 pts) Is the fixed point in (b) stable or unstable? Explain your conclusion.

Because $|a| = 1.013 > 1$, the fixed point is unstable.

(d) (3 pts) In 2010, the population was 30,000. Does the population increase or decrease in next 100 years? Explain your answer.

Because the fixed point is unstable, if $P_0 = 30000$ is larger than the fixed point, then P_n is monotonically increasing.

(e) (3 pts) Assume that the government of Montana wants to stabilize the population of wild bisons based on the population in 2010 (which was 30,000), by changing the number of legal hunting. How many huntings should be allowed?

We want that 30000 is the fixed point.

$$30000 = \frac{b}{1-1.013} \Rightarrow b = -0.013 \cdot 30000 = -390$$

Therefore the hunting of 390 bisons should be allowed.

(8) (5 pts) For a given collection of experimental data $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$, we would like to find a least square line $y = ax + b$ which gives the smallest error with respect to the data. Derive the formula for the *total error* and explain your answer.

For each x_i, $ax_i + b$ is the expected value based on the model, and y_i is the experimental data. So the difference $ax_i + b - y_i$ is the i-th error. To avoid a problem involving sign of each error, we add their squares. Thus

$$\sum_{i=1}^{n}(ax_i + b - y_i)^2$$

is the total error.

- Writing the correct formula $\sum_{i=1}^{n}(ax_i + b - y_i)^2$ of the total error: 3 pts.
- Giving a neat explanation of the formula: + 2 pts.