MATH 1700 MIDTERM EXAM 2 SOLUTION

SPRING 2015 - MOON

(1) Suppose that $P_{n+1} = 5 + \frac{6}{P_n}$ is a price model.

(a) (4 pts) Find all positive fixed prices.

$$x = 5 + \frac{6}{x} \Rightarrow x^2 = 5x + 6 \Rightarrow x^2 - 5x - 6 = 0$$

$$\Rightarrow (x-6)(x+1) = 0 \Rightarrow x = -1, 6$$

Therefore there is only one positive fixed point, $x = 6$.

- Setting up the equation $x = 5 + \frac{6}{x}$ to find fixed points: 1 pt.
- Getting the answer 6 with appropriate steps: 4 pts.

(b) (3 pts) Determine their stability. No cobwebbing is allowed.

$$f(x) = 5 + \frac{6}{x} \Rightarrow f'(x) = -\frac{6}{x^2}$$

$$f'(6) = -\frac{6}{6^2} = -\frac{1}{6}$$

Because $|f'(6)| = |-\frac{1}{6}| < 1$, 6 is locally stable.

- If the computation of $f'(x)$ is incorrect, 1 pt.

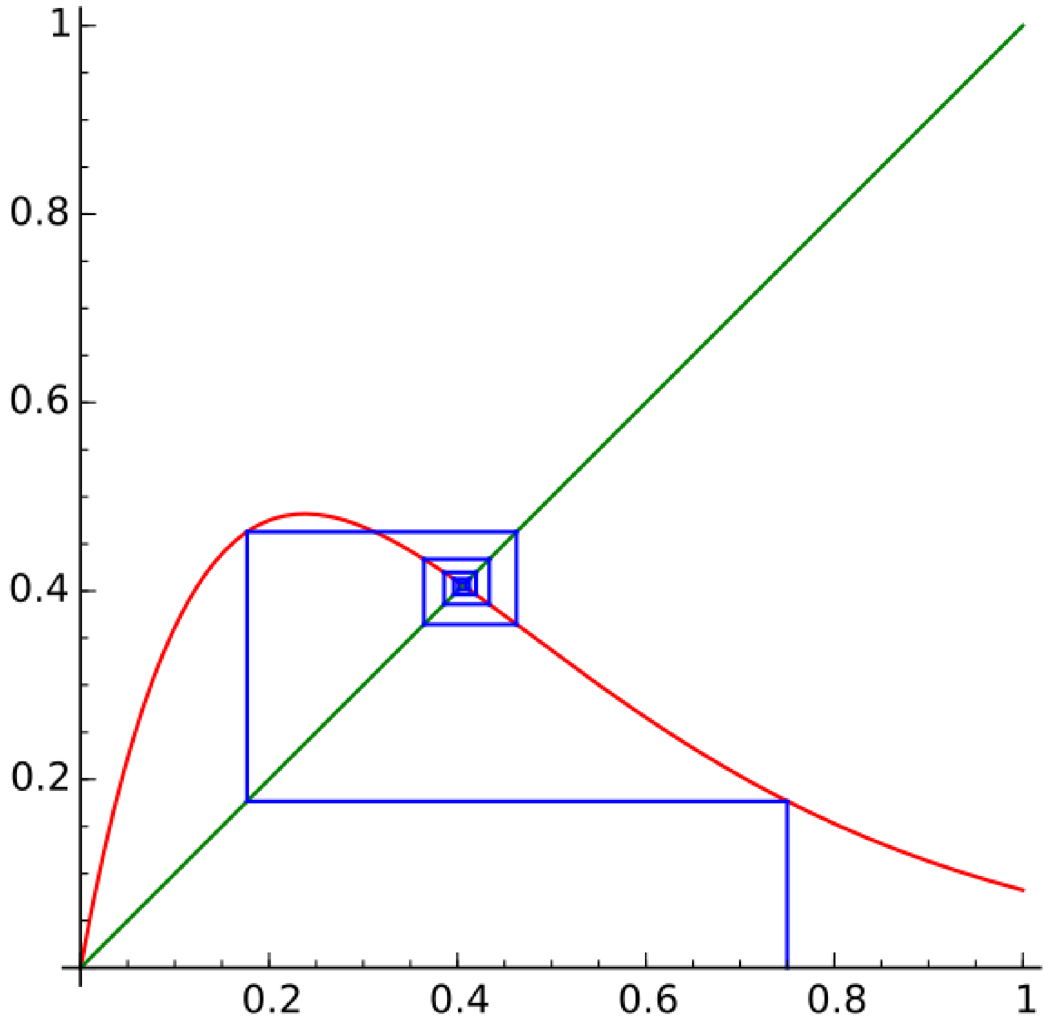
(c) (3 pts) Determine whether there is an oscillation around them or not. No cobwebbing is allowed.

$$f'(6) = -\frac{1}{6} < 0$$

So there is an oscillation around 6.

Date: April 13, 2014.

(2) The following graph is a graph of $y = f(x)$ and $y = x$.



(a) (3 pts) Indicate all fixed points on the graph.

There are two fixed points (0 and $p \approx 0.4$).

(b) (4 pts) Sketch a cobweb graph starting from $x = 0.75$.

(c) (3 pts) Write what we may guess from (b) about the stability of a fixed point.

We may guess that the positive fixed point p is locally stable.

(3) (7 pts) State the Mean Value Theorem.

Suppose that $f(x)$ is continuous on $[a, b]$ and is differentiable on (a, b). Then there is c on $[a, b]$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

- Not mentioning the differentiability on (a, b): -2 pts.
- Not mentioning $a \leq c \leq b$: -2 pts.
- Stating $\dfrac{f(b) - f(a)}{b - a} = f'(c)$: 3 pts.

(4) Let $g(x) = x^2 - 2x$.

(a) (7 pts) Find a 2-cycle of $g(x)$.

$$g(x) = x \Rightarrow x^2 - 2x = x \Rightarrow x^2 - 3x = x(x - 3) = 0 \Rightarrow x = 0, 3$$

So there are two fixed points $x = 0$ and $x = 3$.

$$g(g(x)) = x \Rightarrow g(x^2 - 2x) = x \Rightarrow (x^2 - 2x)^2 - 2(x^2 - 2x) = x$$
$$\Rightarrow x^4 - 4x^3 + 2x^2 + 3x = 0$$

Note that $x^2 - 3x$ must be a divisor of the degree 4 polynomial. By polynomial factorization,

$$x^4 - 4x^3 + 2x^2 + 3x = (x^2 - 3x)(x^2 - x - 1) = x(x - 3)(x^2 - x - 1)$$

and

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

are two additional zeros. Therefore $\left\{\dfrac{1+\sqrt{5}}{2}, \dfrac{1-\sqrt{5}}{2}\right\}$ is a 2-cycle.

- Finding all fixed points $x = 0, 3$: 3 pts.
- Finding the 2-cycle $\left\{\dfrac{1+\sqrt{5}}{2}, \dfrac{1-\sqrt{5}}{2}\right\}$: 7 pts.

(b) (4 pts) Determine the stability of the 2-cycle.

$$g'(x) = 2x - 2$$

$$|g'(\frac{1+\sqrt{5}}{2})||g'(\frac{1-\sqrt{5}}{2})| = |\left(2 \cdot \frac{1+\sqrt{5}}{2} - 2\right)\left(2 \cdot \frac{1-\sqrt{5}}{2} - 2\right)|$$
$$= |(-1 + \sqrt{5}) \cdot (-1 - \sqrt{5})| = 4 > 1$$

Therefore this 2-cycle is unstable.

- Stating the fact that $|g'(p_1)g'(p_2)| <\Leftrightarrow$ stable: 1 pt.

(5) Consider a one parameter family $f_r(x) = rx(3 - x^2)$ of functions with $r > 0$.

(a) (3 pts) Find the interval of stability of the fixed point 0.

$$f_r(x) = 3rx - rx^3 \Rightarrow f_r'(x) = 3r - 3rx^2 = 3r(1 - x^2) \Rightarrow f_r'(0) = 3r$$

$$|f_r'(0)| = 3r < 1 \Leftrightarrow r < \frac{1}{3}$$

Therefore the interval of stability is $(0, \frac{1}{3})$.

(b) (5 pts) Find a positive fixed point $p(r)$ and its interval of existence.

$$f_r(x) = x \Rightarrow rx(3 - x^2) = x \Rightarrow r(3 - x^2) = 1$$

$$\Rightarrow rx^2 + 1 - 3r = 0 \Rightarrow x = \pm\sqrt{\frac{3r - 1}{r}}$$

Therefore the positive fixed point is $p(r) = \sqrt{\frac{3r-1}{r}}$. Note that this is a positive real number only if $3r - 1 > 0$, or equivalently, $r > \frac{1}{3}$. Thus the interval of existence is $(\frac{1}{3}, \infty)$.

- Finding the positive fixed point $p(r) = \sqrt{\frac{3r-1}{r}}$: 3 pts.
- Finding the interval of existence $(\frac{1}{3}, \infty)$: 5 pts.

(c) (4 pts) Find the interval of stability of $p(r)$.

$$f_r'(x) = 3r - 3rx^2 \Rightarrow f_r'(p(r)) = f_r'(\sqrt{\frac{3r-1}{r}}) = 3r - 3r\left(\sqrt{\frac{3r-1}{r}}\right)^2$$

$$= 3r - 3(3r - 1) = 3 - 6r$$

$$|f_r'(p(r))| < 1 \Leftrightarrow |3 - 6r| < 1 \Leftrightarrow \frac{1}{3} < r < \frac{2}{3}$$

So the interval of stability is $(\frac{1}{3}, \frac{2}{3})$.

(d) (3 pts) Find the first bifurcation point r and describe what happens at the point.

When $r = \frac{1}{3}$, $p(r) = p(\frac{1}{3}) = 0$. At $r = \frac{1}{3}$, a stable fixed point 0 becomes unstable, and another positive fixed point $p(r)$ bifurcates from 0.

- Saying that $r = \frac{1}{3}$ is the first bifurcation point: 1 pt.
- Mentioning $p(\frac{1}{3}) = 0$: 1 pt.
- Explaining the stable fixed point 0 becomes unstable, and another stable fixed point $\frac{1}{3}$ bifurcates from 0: 1 pt.

(6) (7 pts) Choose *one* of two models below and explain how the model was derived. In other words, explain each term on the iterative model.

- Logistic population model $P_{n+1} = rP_n \left(1 - \frac{P_n}{C}\right)$
- Contagious disease model $I_{n+1} = I_n - rI_n + sI_n \left(1 - \frac{I_n}{N}\right)$

In the logistic population model, P_n is describing the population of a species in n-th period. r is the initial population growth rate, and C is the capacity of the environment. As the population grows, the growth ratio decreases because of the lack of foods, spaces, and hygiene. To apply this idea, we multiply $1 - \frac{P_n}{C}$, which is a decreasing function in terms of P_n.

In the contagious disease model, I_n is describing the number of people who are infected by a disease in n-th period. The constant r is the rate that the infected people are recovered, and s is a positive constant proportional to the number of newly infected people, and N is the total population. The number of infected people in the next term I_{n+1} is described as $I_n -$ recovered cases $+$ new cases. The second term is rI_n, by the definition of r. The number of newly infected people is proportional to both the number of infected people (I_n) and the number of healthy people ($N - I_n$) because an infection occurs when a healthy person and an infected person meet. Thus the number of new cases is $kI_n(N - I_n)$ for some $k > 0$. If we set $s = kN$, then

$$kI_n(N - I_n) = kNI_n \left(1 - \frac{I_n}{N}\right) = s\left(1 - \frac{I_n}{N}\right).$$

- Stating the meaning of a model correctly: 2 pts.
- Explaining the constants r, C in logistic model or r, s, N in contagious disease model: 2 pts.
- Describing why the model is reasonable: 3 pts.