Computer project 1 Solution

The goal of this project is to investigate the contagious behavior of a disease, which is modeled by the following iterative non-linear model

$$I_{n+1} = f(I_n) = I_n - rI_n + sI_n^2\left(1 - \frac{I_n}{N}\right),$$

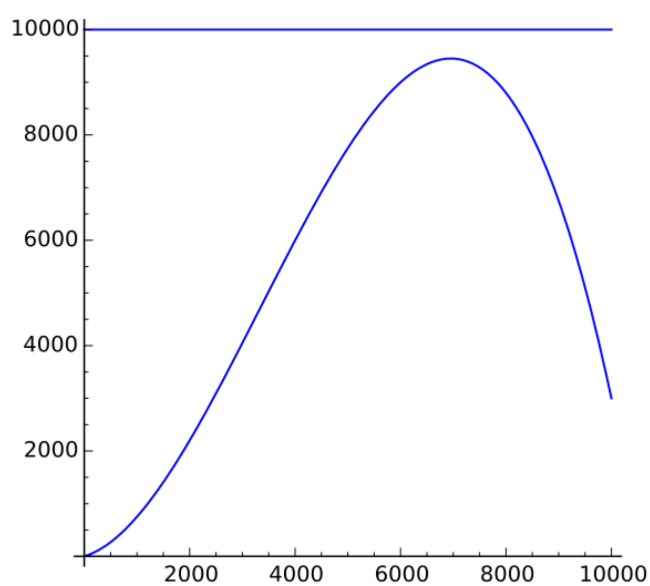
where I_n is the number of infected people at n-th week, N is the total population, r is the recovery rate, and s is a positive constant. Note that this model is different from the non-linear infection model we've discussed in class.

1. Suppose that the recovery rate r is 0.7 and the total population is 10000. Find the condition for s that $f(x)$ defines a dynamical system (note that the image of f must be in the domain of f).

   For this model, the domain is $[0, 10000]$. So $I_{n+1} = f(I_n)$ is an unbreakable dynamical system if the image of f is in $[0, 10000]$.

   If we plot a general graph of $f(x)$ (with an appropriate s), the graph looks like the following:

```
plot(x-0.7*x+0.0005*x^2*(1-x/10000),(x,0,10000),aspect_ratio=1)
+plot(10000,(x,0,10000))
```



So the image of f is in $[0, 10000]$ if the maximum occurring when x is around 7000 is less than 10000. By sketching graphs with various s, we can conclude that if s is in $(0, 0.000537]$, then f defines an unbreakable dynamical system.

2. In addition to conditions in 1, suppose that $s = 0.00051$. Find all fixed points.

```
solution_set = solve(x-0.7*x+0.00051*x^2*(1-x/10000) == x,x)
for s in solution_set:
    print float(s.right_hand_side())
1642.24631578
8357.75368422
0.0
```

Thus we can find three fixed points 1642.24631578, 8357.75368422, and 0.

3. By making a table of I_n for $0 \le n \le 20$ with various I_0, investigate the stability of each fixed point.

```
I = [1500]
for j in range(20):
    I.append(I[j]-0.7*I[j]+0.00051*I[j]^2*(1-I[j]/10000))
for j in range(21):
    print "I_%(index)s = %(value)f" % {"index" : j, "value" : I[j]}
```

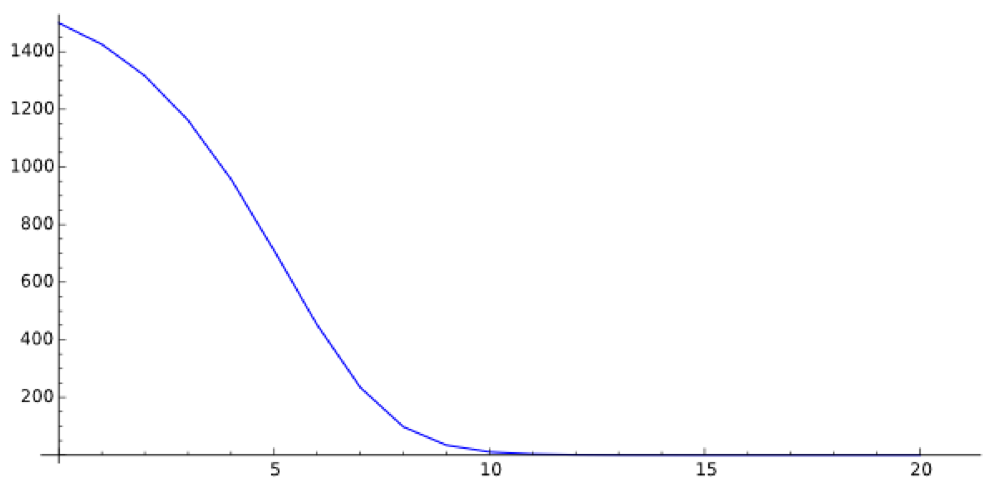
I_0	1500	2000	8300	8400
I_1	1425.37500000000	2232.00000000000	8462.76300000000	8277.69600000000
I_2	1316.08417457192	2643.23925043200	8153.64274782236	8501.95662730300
I_3	1161.92725240351	3414.35042521862	8706.30813665211	8073.04387281909
I_4	957.113262921824	4939.78484089634	7613.03334108123	8826.87620489286
I_5	709.611820910897	7779.24766367345	9339.47817518724	7309.58865340380
I_6	451.469965664620	9187.79639215009	5740.18699788432	9524.07102553501
I_7	234.698738802755	6253.03442182683	8880.40384458360	5058.91841808502
I_8	97.8428762156553	9347.81836862655	7167.06993255011	7966.90087564745
I_9	34.1874390796360	5710.77150765505	9571.58034941804	8971.30765883856
I_{10}	10.8502721901360	8847.32707986770	4873.21037134714	6913.86823634074
I_{11}	3.31505799775875	7255.70878700352	7671.31100358990	9597.78021883641
I_{12}	1.00012024219632	9544.88977991526	9290.48674349171	4768.95722453807
I_{13}	0.300546144294911	4978.06874490534	5910.40233437492	7498.12452903822
I_{14}	0.0902099091762131	7840.33616131185	9059.04829295493	9423.10353581813
I_{15}	0.0270671230075581	9122.67814067177	6655.95885287066	5439.42913578732
I_{16}	0.00812051054212153	6460.49612906376	9552.28479882002	8513.53341121895
I_{17}	0.00243618679338180	9472.47312686494	4949.15203129925	8048.77307863403
I_{18}	0.000730859064866908	5255.76563003046	7794.26184761633	8861.32986951088
I_{19}	0.000219257991879089	8260.29648249697	9172.26502788581	7218.41018965167
I_{20}	0.0000657774220815002	8532.00909857338	6303.20282852698	9557.25787004343

From the table above, we can conclude that 0 is stable, 1642.24631578 and 8357.75368422 are unstable.

4. In 3, you can see that the long-term behavior of the solution heavily depends on the initial condition I_0. Find the *threshold* level, that is, the critical initial condition I such that the behavior of I_n with $I_0 < I$ is completely different from that with $I_0 > I$. Sketch two time series graphs of the solution I_n with $I_0 < I$ and $I_0 > I$ and explain the graph in words.

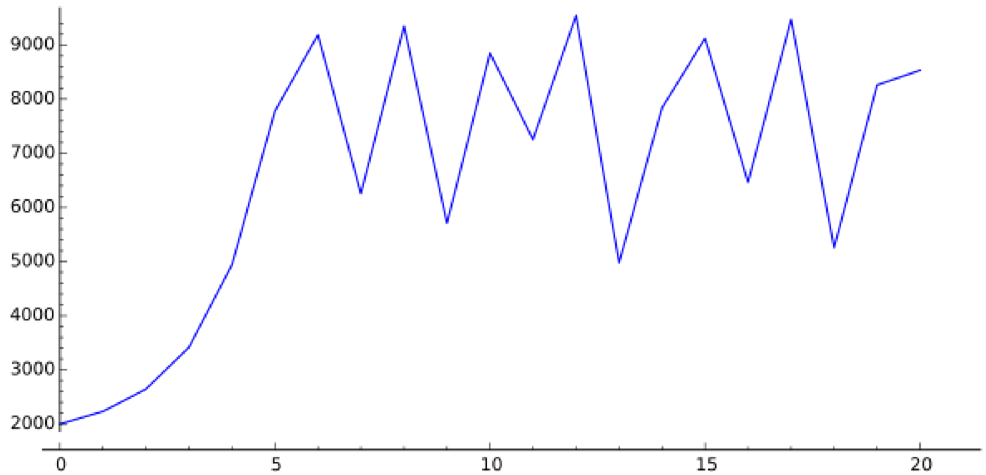
   In 3, if $I_0 = 1500$ the solution is approaching 0. But if $I_0 \geq 2000$, the solution does not approach any fixed points. By plug in several numbers between 1500 and 2000 for the above code in 3, the threshold level I is the smaller fixed point $p = 1642.24631578$. If the initial condition is less than p, then the solution converges to 0. But if $I_0 > I = p$, then the long-term behavior of the solution is unpredictable. The following time series graphs are showing the threshold effect.

```
I = [1500]
for j in range(20):
    I.append(I[j]-0.7*I[j]+0.00051*I[j]^2*(1-I[j]/10000))
tsgraph = finance.TimeSeries(I)
```



The time series graph for $I_0 = 1500$

```
I = [2000]
for j in range(20):
    I.append(I[j]-0.7*I[j]+0.00051*I[j]^2*(1-I[j]/10000))
tsgraph = finance.TimeSeries(I)
```



The time series graph for $I_0 = 2000$

- Each question is 5 points, and the overall completeness including formatting is 10 points. Remember that the output is a summarized report, not a bulk of codes or answers.
- If you provide a complete calculation or appropriate reasoning including Sage code, you get 5 points. If the reasoning is incomplete, then you can get 3 points.
- To organize your conclusion, you can freely use any tools such as MS words, Pages, LaTeX, or even handwriting, but you have to provide a neatly organized result.