Computer project 2

This project must be submitted on paper. Keep in mind that the output should be a research report - you should provide answers and (mathematical/experimental) evidences. Feel free to use MS words, Pages, LaTeX, even handwriting, but include supporting evidences such as your sage codes, graphs and data. For the below problems, you *don't* need to find the precise answer unless there is a special indication. By using a computer algebra system, find the best solution you can find. The due date is April 16th.

The goal of this project is to analyze the bifurcation of one parameter family of population models. Consider a one parameter family of non-linear population model

$$x_{n+1} = rx_n e^{-x_n},$$

where x_n is the population of a species of ground beetle at n-th year (the unit is million), and $r > 0$ is the growth rate. This iterative dynamical system is given by a function $f_r(x) = rxe^{-x}$.

1. (Solve this problem without using computer algebra system) Note that 0 is a fixed point for every $r > 0$. Find the interval of stability.

$$f_r'(x) = re^{-x} + rx(-1)e^{-x} = r(1-x)e^{-x}$$

$$f_r'(0) = r(1-0)e^0 = r$$

$$|f_r'(0)| < 1 \Leftrightarrow r = |r| < 1$$

   Therefore the interval of stability of 0 is $(0, 1)$.

2. (Solve this problem without using computer algebra system) Find the interval of existence/stability for a positive fixed point.

$$f_r(x) = x \Rightarrow rxe^{-x} = x \Rightarrow x(re^{-x} - 1) = 0 \Rightarrow re^{-x} = 1$$

$$\Rightarrow e^{-x} = \frac{1}{r} \Rightarrow -x = \ln\frac{1}{r} = -\ln r \Rightarrow x = \ln r$$

   So the nonnegative fixed point is $\ln r$ and it is positive only if $r > 1$. Therefore the interval of existence is $(1, \infty)$.

$$f_r'(\ln r) = r(1-\ln r)e^{-\ln r} = r(1-\ln r)\frac{1}{r} = 1 - \ln r$$

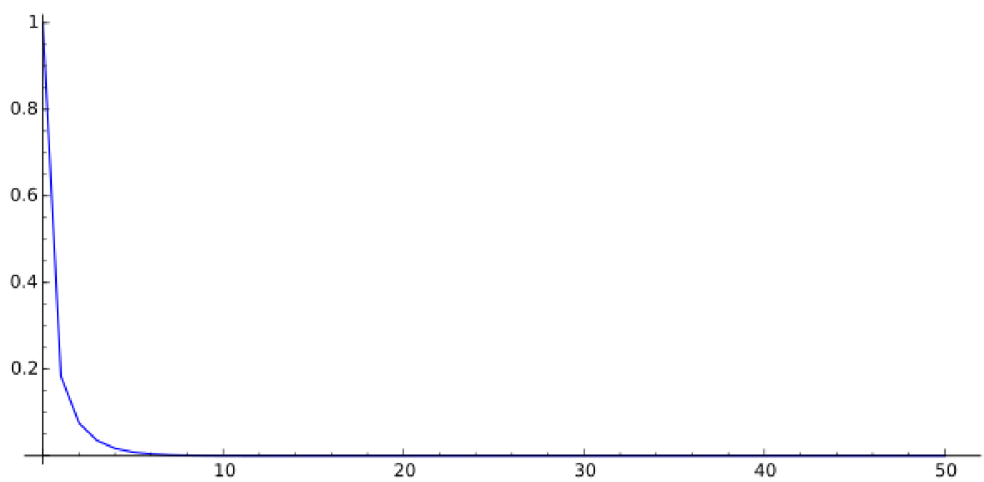
$$|f_r'(\ln r)| < 1 \Leftrightarrow |1 - \ln r| < 1 \Leftrightarrow 0 < \ln r < 2 \Leftrightarrow 1 < r < e^2$$

   Therefore the interval of stability is $(1, e^2)$.

3. For each interval of stability (of 0, a positive fixed point, a stable 2-cycle) choose one value of r and sketch a time series graph (for 50 iterations) which shows a stable behavior.

   If $r = 0.5$, the time series graph with $x_0 = 1$ is:

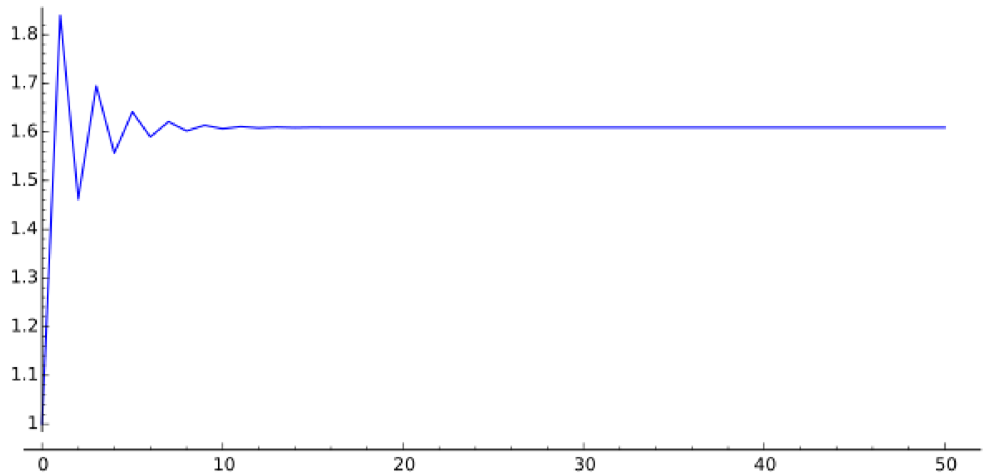
```
x = [1.0]
for i in range(50):
    x.append(0.5*x[i]*exp(-x[i]))
tsgraph = finance.TimeSeries(x)
tsgraph.plot()
```



   It shows the stability of the fixed point 0.

   When $r = 5$, the time series graph with $x_0 = 1$ is:

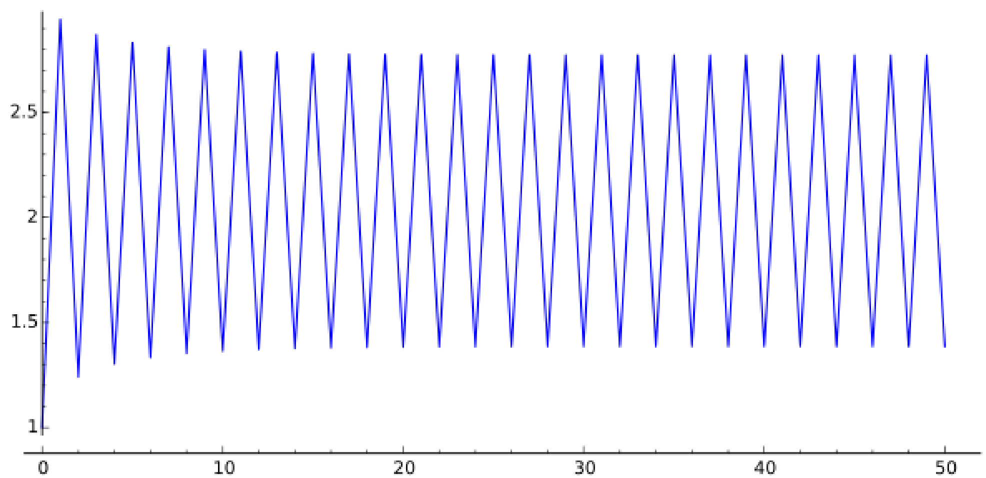
```
x = [1.0]
for i in range(50):
    x.append(5*x[i]*exp(-x[i]))
tsgraph = finance.TimeSeries(x)
tsgraph.plot()
```



This graph shows the stability of the positive fixed point $\ln 5 \approx 1.6094$.

Finally, if $r = 8$, the times series graph with $x_0 = 1$ is:

```
x = [1.0]
for i in range(50):
    x.append(8*x[i]*exp(-x[i]))
tsgraph = finance.TimeSeries(x)
tsgraph.plot()
```

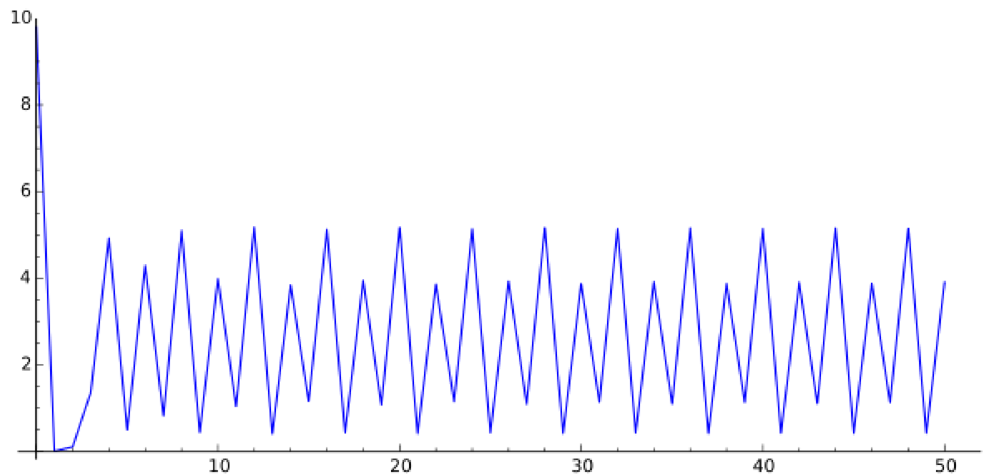


We can see a stable 2-cycle.

4. It is known that *Carabus auratus*, the golden ground beetle has the growth rate 14.09. In 2013, because of an unusual weather, its population in Vanoise National Park in France was increased to 9.8 million. Discuss the expected population in next 50 years.

```
x = [9.8]
print "x_%(index)s = %(value)f" % {"index" : 0, "value" : x[0]}
for i in range(50):
    x.append(14.09*x[i]*exp(-x[i]))
    print "x_%(index)s = %(value)f" % {"index" : i+1, "value" : x[i+1]}
tsg_x = finance.TimeSeries(x)
tsg_x.plot()
```

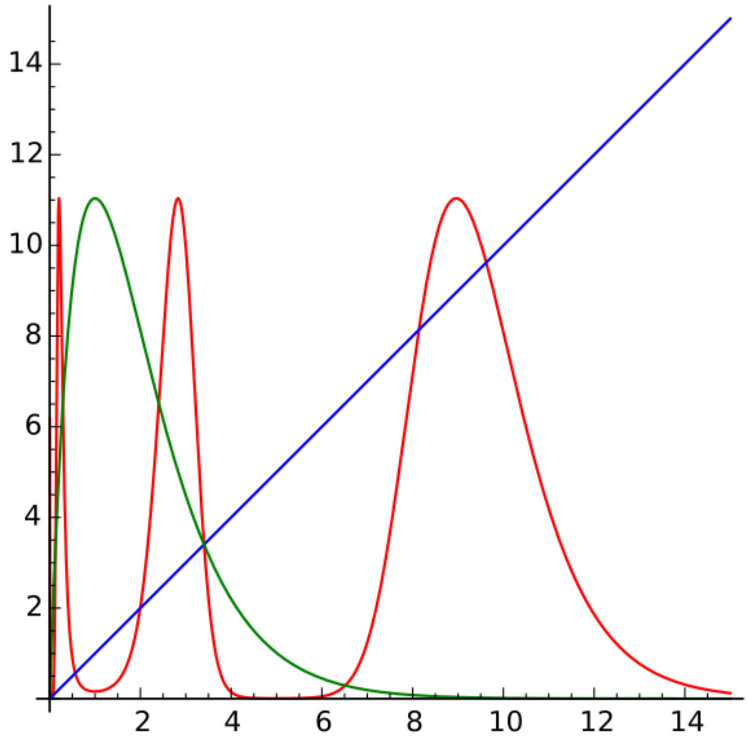
The following is the time series graph we can obtain:



In the next year, most of the beetles disappear. But from 2014, it will become increase, and from 2021, it shows a period 4 cycle.

5. When the growth ratio is sufficiently large, this population model shows a chaotic behavior. Find r so that the corresponding population model is chaotic. (Hint: If there is a 3-cycle, then $x_{n+1} = f_r(x_n)$ is chaotic.)

```
f = 30*x*exp(-x)
g = f(f(f))
plot(g,0,15,rgbcolor='red',aspect_ratio=1)+
  plot(x,0,15,rgbcolor='blue')+plot(f,0,15,rgbcolor='green')
```

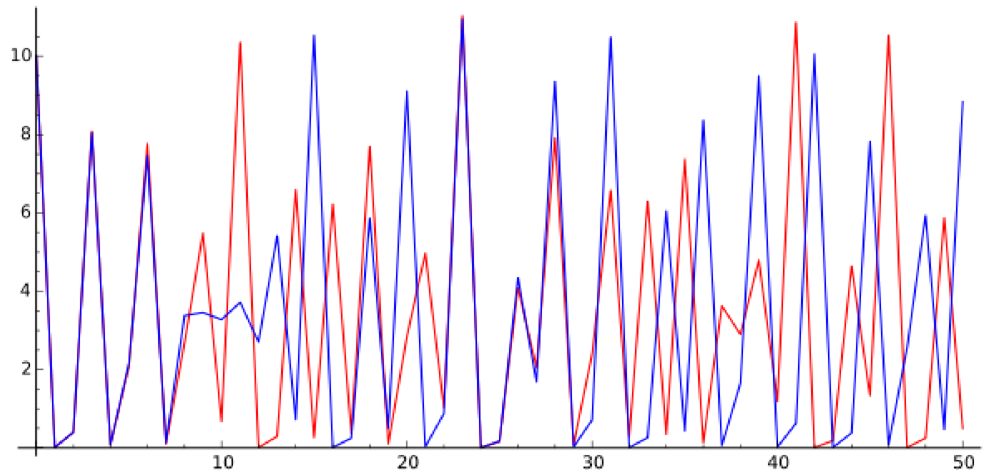


Take $r = 30$. The green graph is the graph of $f(x) = 30xe^{-x}$, the red graph is that of $g = f^3$. The intersection of $y = g(x)$ and $y = x$ can be a point on a 3-cycle or a fixed point. But if it is a fixed point, it is also on the graph of $y = f(x)$. Because there are some points on the intersection of the line $y = x$ and the graph of $y = g(x)$ but not lying on the graph of $y = f(x)$, those points form a 3-cycle. Because it has a 3-cycle, $x_{n+1} = f(x_n)$ is chaotic.

6. Take r in the above question. Sketch two time-series graphs with very close initial conditions showing the sensitive dependence on initial conditions.

   We will take to initial conditions 10.0 and 10.01.

```
x1 = [10.0]
x2 = [10.01]
for i in range(50):
    x1.append(30*x1[i]*exp(-x1[i]))
    x2.append(30*x2[i]*exp(-x2[i]))
tsg_x1 = finance.TimeSeries(x1)
tsg_x2 = finance.TimeSeries(x2)
plot(tsg_x1,rgbcolor='red') + plot(tsg_x2,rgbcolor='blue')
```



The above time series graph shows that after 7th iteration, two solutions are very different.

- Each question is 4 points, and the overall completeness including formatting is 6 points. Remember that the output must be a summarized report, not a bulk of codes or answers.
- If you provide a complete calculation or appropriate reasoning including Sage code, you get 4 points. If the reasoning is incomplete, then you can get 2 points.
- Unless you provide computer codes and the output such as time series graph obtained by running the code, you cannot get the credit.